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Non-deterministic conditionals and transparent truth

Abstract. Theories where truth is a naive concept fall under the following dilemma: either the theory is subject to Curry's Paradox, which engenders triviality, or the theory is not trivial but the resulting conditional is too weak. In this paper we explore a number of theories which arguably do not fall under this dilemma. In these theories the conditional is characterized in terms of (infinitely-valued) non-deterministic matrices. These non-deterministic theories are similar to infinitely-valued Łukasiewicz logic in that they are consistent and their conditionals are quite strong. The difference is the following: while Łukasiewicz logic is ω-inconsistent, the non-deterministic theories might turn out to be ω-consistent.

Keywords: Naive truth theory, Łukasiewicz logic, Curry's paradox, non-deterministic semantics, ω-inconsistency.

1. Introduction

In this paper we want to address the problem of finding a strong conditional connective for a naive truth theory[1]. This problem is not really new. Usually, theories where truth is treated as a naive concept fall under the following dilemma: either the theory is subject to Curry's Paradox, which engenders triviality, or the theory is not trivial but the resulting conditional is too weak[2]. Recently there have been many attempts to avoid this dilemma by the introduction of rather complicated conditionals (see for example [3], [5], [6], and [12]).

One relatively familiar and uncomplicated conditional which does not fall under the dilemma is the conditional of Łukasiewicz continuum-valued logic. However, Hartry Field [6], p.94 claims that:

Presented by **Name of Editor**; *Received* December 1, 2005

[1]By a *naive truth theory* we mean a theory that contains all instances of the schema $Tr\ulcorner\phi\urcorner \leftrightarrow \phi$ (where $\ulcorner\phi\urcorner$ is a name for the sentence ϕ). This is usually taken to be different from a *transparent truth theory*, a theory where $Tr\ulcorner\phi\urcorner$ and ϕ are everywhere intersubstitutable. However, in the non-deterministic theories we introduce below, this distinction will be negligible

[2]Some substructural theories of truth do not fall under this dilemma, but here we will only consider theories with a consequence relation that satisfies all the usual structural properties.

> (...) the clear inadequacy of the continuum-valued semantics for languages with quantifiers should not blind us to its virtues in a quantifier-free context. Indeed, one might well hope that some simple modification of it would work for languages with quantifiers. In fact, this does not seem to be the case: major revisions in the approach seem to be required.

Now, we do not know exactly what Field means by 'major revisions', but here we will consider several close cousins of Łukasiewicz logic and argue that most of them are at least not clearly inadequate. Although this can be done proof-theoretically -by analyzing which axioms and rules must be satisfied by the target conditional- we employ a semantic approach. In particular, we will use non-deterministic matrices to obtain (relatively) strong subtheories of Łukasiewicz continuum-valued logic.

The rest of the paper is structured as follows. As the technique of non-deterministic matrices might be unfamiliar for philosophers interested in semantic paradoxes, the next section gives a brief and sketchy tutorial on the topic. In Section 3, after showing the inadequacy of finitely-valued Łukasiewicz logic (whether deterministic or not), we present the well-known continuum-valued version of Łukasiewicz logic. Section 4 contains several ways of making this logic non-deterministic and shows how strong the resulting theories are. Section 5 contains some speculative remarks on whether the non-deterministic theories we consider are ω-inconsistent, and section 6 shows how it is possible to define a determinately operator in these theories. Section 7 compares the present proposal to a similar but slightly different approach developed recently by Andrew Bacon in [2] and contains some closing remarks.

2. Non-deterministic matrices

The idea of a non-truth-functional connective is quite old and well-known. Recently, though, this idea has been studied by using what is sometimes called 'non-deterministic matrices'. This formal tool has been rigorously developed by those -specially computer scientist- who wish to study a number of properties of proof systems from a semantic point of view[3]. Intuitively, in a non-deterministic framework there is at least one connective such that you cannot completely determine the value of a compound formula involving

[3] A very complete introduction to the topic together with a brief survey on possible applications of non-deterministic matrices can be found in [1]. In any case, we think non-deterministic matrices have not been sufficiently explored as a tool to study *semantic paradoxes* (nor, for that matter, to study other kinds of philosophical puzzles).

that connective even if you know the values of all the atomic formulas of the language. We can give a more formal definition in the following way:

DEFINITION 2.1. *($\mathcal{ND}Matrix$) A* non-deterministic matrix *for a language $\mathcal{L}$ is a tuple $\mathcal{M} = <\mathcal{V}, \mathcal{D}, \mathcal{O}>$, where:*

- *$\mathcal{V}$ is a non-empty set of truth values,*
- *$\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and*
- *$\mathcal{O}$ is a set of functions such that for every n-ary connective $\diamond$ in $\mathcal{L}$, there is a corresponding function $\diamond^{\mathcal{M}}$ in $\mathcal{O}$ such that $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{P}(\mathcal{V}) - \varnothing$.*[4]

The interesting part of the definition has to do with the set $\mathcal{O}$ of functions for the non-deterministic connectives. In a deterministic matrix, for each n-ary connective $\diamond$ in $\mathcal{L}$, there is a corresponding function $\diamond^{\mathcal{M}}$ such that $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{V}$. In the case of non-deterministic connectives, the co-domain of the corresponding function is the set of sets of values $\mathcal{P}(\mathcal{V}) - \varnothing$, rather than the set of values $\mathcal{V}$.

Also notice that deterministic matrices are a special case of non-deterministic matrices. More specifically, for each n-ary connective $\diamond$ in a deterministic matrix $\mathcal{M}$ which is interpreted as a function $\diamond^{\mathcal{M}}$: $\mathcal{V}^n \longrightarrow \mathcal{V}$, we can build a non-deterministic matrix $\mathcal{M}'$ where that connective can be taken as a function that only outputs singleton values, that is, $\diamond^{\mathcal{M}'}$: $\mathcal{V}^n \longrightarrow \{\mathcal{A} \subseteq \mathcal{V}: |\mathcal{A}| = 1\}$. By doing this we obtain a non-deterministic matrix with connectives that mimic the behavior of the deterministic connectives.

It is straightforward how to characterize the notion of a *valuation* for a non-deterministic matrix:

DEFINITION 2.2. *(Valuation) Let $Form_{\mathcal{L}}$ denote the set of formulae of the language $\mathcal{L}$. A* valuation *in $\mathcal{M}$ is a function v: $Form_{\mathcal{L}} \longrightarrow \mathcal{V}$ such that for each n-ary connective $\diamond$ of $\mathcal{L}$, the following holds for all $\phi_1, ..., \phi_n \in Form_{\mathcal{L}}$: $v(\diamond(\phi_1, ..., \phi_n)) \in \diamond^{\mathcal{M}}(v(\phi_1), ..., v(\phi_n))$*

Notice that since $\diamond^{\mathcal{M}}(v(\phi_1), ..., v(\phi_n))$ gives a set of values rather than a single value, we use $\in$ instead of $=$ in the previous definition. The concepts of $satisfaction$ and $validity$ can be defined in the usual way.

To see how this in fact works, let's look at an example of a non-deterministic matrix that might be relevant for the study of semantic paradoxes.

[4]The reason for excluding the empty set is that it is not straightforward how to compute the value of compound formulae where at some step of the computation we have as input the empty set.

EXAMPLE 2.3. *Let $\mathcal{L}$ be a propositional language with one unary connective $\neg$ and two binary connectives $\vee$ and $\wedge$. Let $\mathcal{M}_1 = < \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 >$, where:*

- $\mathcal{V}_1 = \{1, 0\}$,
- $\mathcal{D}_1 = \{1\}$, *and*
- $\mathcal{O}_1$ *is defined in the following way:*

	$\neg^{\mathcal{M}_1}$
1	{*0*}
0	{*1,0*}

		$\vee^{\mathcal{M}_1}$
1	*1*	{*1*}
1	*0*	{*1*}
0	*1*	{*1*}
0	*0*	{*0*}

		$\wedge^{\mathcal{M}_1}$
1	*1*	{*1*}
1	*0*	{*0*}
0	*1*	{*0*}
0	*0*	{*0*}

A reason for employing this matrix is that it is compatible with a theory of transparent truth. It is well-known that the paracomplete three-valued logic K_3 can support a transparent truth predicate. But it is easy to see that every K_3-countermodel can be turned into an $\mathcal{M}_1$-countermodel by replacing all assignments of $\frac{1}{2}$ by 0, and leaving everything else untouched. This means that $\mathcal{M}_1$ is a sublogic of K_3. So $\mathcal{M}_1$ is a (paracomplete two-valued!) consistent non-deterministic matrix with a transparent truth predicate[5].

From this example it should be clear that taking a deterministic matrix and making it non-deterministic (possibly) weakens the resulting logic. This makes sense: a non-deterministic matrix considers more valuations than a deterministic one, at least *ceteris paribus*. So if we are in the business of solving paradoxes by weakening logic, non-deterministic matrices might be a good tool to see what sort of paradox-free logics we can obtain[6].

3. Łukasiewicz logic

Multivalued logics such as K_3 and LP can arguably be considered as plausible solutions to the Liar Paradox. However, the material conditional exhibits an odd behavior in these logics: Modus Ponens ($\phi, \phi \supset \psi \nvDash_{LP} \psi$) does not holds in LP, while Identity ($\nvDash_{K_3} \phi \supset \phi$) does not hold in K_3. Both Field [6] and Beall [5] have worked on supplementing such theories with a suitable conditional. However, Curry's paradox makes this task quite complicated.

[5]The logic characterized by this matrix is the conditional-free fragment of the logic known as $CLaN$. Notice also that the dual of the logic characterized by $\mathcal{M}_1$ is the conditional-free fragment of the paraconsistent logic $CLuN$, which is a sublogic of the paraconsistent logic LP, dual to K_3. For more details on these logics see [4].

[6]For more on the project of using non-deterministic matrices to deal with semantic paradoxes see [11].

Is there any other way of supplementing these logics with a suitable conditional? That depends, of course, on what we take a suitable conditional to be. One option is to say that a suitable conditional is one that satisfies certain principles and rules of inference. Another option is to impose general constraints regarding the way in which a valuation should behave with respect to the conditional. These are not mutually exclusive approaches. We could impose constraints on the valuations in such a way that the conditional validates the principles and rules we want. Nonetheless, it might be that in certain contexts one of the approaches is more illuminating than the other.

If we are working in a *linearly ordered* space of values[7], it seems useful to embrace the second approach and to say that a conditional $\rightarrow$ is *suitable* if it is not subject to Curry's Paradox and it satisfies the following constraints:

1. If $v(\phi) \leq v(\psi)$, then $v(\phi \rightarrow \psi) \in \mathcal{D}$
2. If $v(\phi) > v(\psi)$, then $v(\phi \rightarrow \psi) \in \mathcal{V} - \mathcal{D}$

Can a transparent truth theory be supplemented with a suitable conditional in this sense? Unfortunately, for any finitely-valued linearly-ordered matrix, regardless of whether it is deterministic or not, the following can be proved[8]:

Theorem 3.1. *No n-valued linearly ordered matrix containing a transparent truth predicate can have a suitable conditional, provided that for each non-designated value i there is a sentence ϕ such that $v(\phi) = i$*[9].

Proof. Assume that $\mathcal{V}$ has n elements. Since the values in $\mathcal{V}$ are linearly ordered by some relation $<$, we can list them as follows: $r_1, ..., r_n$, where $r_1 < .... < r_n$ and $\varnothing \neq \mathcal{D} \subseteq \{r_2, ..., r_n\}$. Since $\mathcal{D}$ is finite, there is a greatest non-designated value $r_i \notin \mathcal{D}$. Now consider a sentence γ such that γ is $Tr\ulcorner\gamma\urcorner \rightarrow \phi$, where $v(\phi) = r_i$. Now we reason thusly: if $v(\gamma) \in \mathcal{D}$, then $v(\gamma) > v(\phi)$. So by constraint 2, $v(\gamma) \in \mathcal{V} - \mathcal{D}$; if $v(\gamma) \in \mathcal{V} - \mathcal{D}$, then $v(\gamma) \leq v(\phi)$. So by constraint 1, $v(\gamma) \in \mathcal{D}$. Either way, we have a contradiction. ∎

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[7]We will assume further that the space of values satisfies the following condition: *for all x and for all y, if $x \in \mathcal{D}$ and $y \in \mathcal{V} - \mathcal{D}$, then $x > y$.*

[8]We assume, as usual, that the language we are working with has some way to talk about itself. More specifically, for each formula ϕ of the language there is a term $\ulcorner\phi\urcorner$ that is the name of (the code of) that formula. So, if the language contains a truth predicate, we can construct a Liar sentence λ such that λ is $\neg\, Tr\ulcorner\lambda\urcorner$, a Curry sentence δ such that δ is $Tr\ulcorner\delta\urcorner \rightarrow \bot$, and so on.

[9]In fact, we can prove that there is such a sentence, so we can dispose of this assumption. However, the proof is simpler this way.

So the problem with finite non-deterministic matrices is that you can use the greatest non-designated value to construct a version of Curry's Paradox[10]. With infinite matrices the problem does not necessarily arise. There might be infinitely many increasing non-designated values, so perhaps there is no greatest non-designated value. However, there is a different problem with infinitely-valued theories that include a naive truth predicate. The best such theory in the market is Łukasiewicz's theory $Ł_\infty$[11], which can be semantically characterized as follows (see [8] for more details on $Ł_\infty$):

DEFINITION 3.2. *(Łukasiewicz logic $Ł_\infty$) Let $Ł_\infty$ be the theory characterized by the matrix $< d, I, \mathcal{V}, \mathcal{D}, \mathcal{O} >$, where*

- *d is a non-empty set,*
- *I is an interpretation function for the non-logical vocabulary,*
- $\mathcal{V} = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$,
- $\mathcal{D} = \{1\}$, *and*
- *$\mathcal{O}$ is defined in the following way:*
  - $v(\neg\phi) = 1 - v(\phi)$,
  - $v(\phi \vee \psi) = max(v(\phi), v(\psi))$,
  - $v(\exists x\phi) = sup\{v'(\phi) : v' \textit{ is an } x\textit{-variant of } v\}$, *and*
  -
$$v(\phi \to \psi) = \begin{cases} 1 & \textit{if } v(\phi) \leq v(\psi) \\ 1 - (v(\phi) - v(\psi)) & \textit{otherwise} \end{cases}$$

We obtain the theory $Ł^+_\infty$ by considering only those valuations that in addition to this satisfy $v(Tr\ulcorner\phi\urcorner) = v(\phi)$, for every ϕ. This theory has some attractive properties, specially regarding the truth predicate and the conditional. For instance, like K_3 and LP, the Liar and other problematic sentences receive the value $\frac{1}{2}$, but unlike K_3 both Identity and all Tr-biconditionals are valid in $Ł^+_\infty$, and unlike LP, Modus Ponens is valid in $Ł^+_\infty$. Also, it is possible to provide a weakly complete axiomatization for its

[10]See [9] for a different version of this result not involving non-deterministic matrices. Restall's theorem is in a way stronger than what we have just proved, since it also applies to some non-linearly ordered space of values. However, in a different sense, it is weaker, since it only considers deterministic matrices.

[11]Usually, if theorists want to stress that the truth predicate is in the language, this goes under the label $Ł_\infty Tr$, and moreover, if there is a base syntax theory present such as Peano Arithmetic, it becomes $Ł^{PA}_\infty Tr$. To ease the notation we will use $Ł^+_\infty$ for the theory with the truth predicate plus some sort of naming system.

propositional Tr-free fragment. More specifically, all Tr-free $Ł_\infty$-tautologies and inferences with finitely many premises are provable from the following four axioms (together with the rule of Modus Ponens):

$\phi \to (\psi \to \phi)$
$(\neg\psi \to \neg\phi) \to (\phi \to \psi)$
$(\phi \to \psi) \to ((\chi \to \phi) \to (\chi \to \psi))$
$((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$

The problem is that $Ł^+_\infty$ has some unpleasant properties as well. First, the previous axiomatization is only *weakly* complete. There are semantically valid Tr-free $inferences$ that are not provable in it.

Secondly, one way to extend this theory to a first-order language is by introducing the following two axioms (and the rule of Generalization):

$\forall x\phi(x) \to \phi(t)$ (where t is free for x in ϕ)
$\forall x(\phi \to \psi) \to (\phi \to \forall x\psi)$ (where x is not free in ϕ)

However, once we do so, the theory is not even weakly complete[12]. There are some first-order semantically valid Tr-free *sentences* which are not provable in this axiomatization.

These two properties are not really disturbing if you do not care much about proof-theory. However, there is a third unpleasant property: $Ł^+_\infty$ is ω-inconsistent.

Definition 3.3. *(ω-inconsistency) We say that a theory $\mathcal{T}$ is* ω-inconsistent *if and only if for some formula $\phi(x)$ and each object o, $\mathcal{T} \vDash \phi[\boldsymbol{o}/x]$ but $\mathcal{T} \vDash \exists x \neg\phi(x)$ (where $\boldsymbol{o}$ is a name for o)*[13].

Assuming, for instance, that the base theory of $Ł^+_\infty$ is Peano arithmetic, the following can be proved:

Theorem 3.4. (See [10], [7], [2]) *$Ł^+_\infty$ is ω-inconsistent*[14].

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[12] [8] refers the reader to a proof by Scarpellini.

[13] We should point out that being ω-inconsistent is different from being inconsistent in ω-logic (i.e. from lacking an ω-model). A theory is inconsistent in ω-logic if the theory together with the ω-rule is inconsistent. A theory that is consistent in ω-logic is ω-consistent, but the converse might fail.

[14] In addition to this, in [7] it is proved that adding compositional axioms for truth to $Ł^+_\infty$ makes the theory inconsistent, and not just ω-inconsistent. More precisely, let $Sent_{Ł^+_\infty}(x)$ be a predicate holding of all and only the (names of) sentences of $Ł^+_\infty$, and

So a question naturally arises: are there interesting subtheories of Ł$_\infty$ with a strong conditional which are not ω-inconsistent?[15]

4. Making Łukasiewicz logic non-deterministic

The first logic we will discuss is $\mathcal{ND}$Ł$^+_\infty$, which is like Ł$^+_\infty$ except for this:

$$v(\phi \to \psi) \in \begin{cases} \{1\} & \text{if } v(\phi) \leq v(\psi) \\ \mathcal{V} - \mathcal{D} & \text{otherwise} \end{cases}$$

Łukasiewicz's conditional is deterministic, $\mathcal{ND}$Ł$^+_\infty$'s conditional is not[16]. Notice that $\mathcal{ND}$Ł$^+_\infty$'s conditional flows very naturally from the two constraints imposed above on any suitable conditional. Here is an incomplete list of principles and inferences which are valid in this theory (in some cases the names are somewhat arbitrary):

$\phi \to \psi, \phi \vDash \psi$	(Modus Ponens)
$\vDash \phi \to \phi$	(Identity)
$\vDash \neg\neg\phi \to \phi$	(Double Negation)
$(\phi \to \psi) \wedge \neg\psi \vDash \neg\phi$	(Modus Tollens)
$\vDash (\phi \wedge (\psi \vee \chi)) \to (\phi \wedge \psi) \vee (\phi \wedge \chi)$	(Distribution)
$\vDash \phi \to (\phi \vee \psi)$	(Disj. Intr.)
$\vDash (\phi \wedge \psi) \to \phi$	(Conj. Elim.)
$\vDash (\phi \to \psi) \vee (\psi \to \phi)$	(Connectivity)
$\phi \vDash \psi \to \phi$	(Positive Weakening)
$\neg\phi \vDash \phi \to \psi$	(Explosion)
$(\phi \to \psi) \wedge (\psi \to \chi) \vDash \phi \to \chi$	(Weak Transitivity)
$\neg\phi \vee \psi \vDash \phi \to \psi$	(Material Conditional)

The problem with this theory is that the conditional is still too weak, as

let $\dot\vee$ be a function that when applied to the codes of two formulas gives the code of their disjunction. It can be shown that Ł$^+_\infty$ already validates every instance of axiom-schemas such as $Tr\ulcorner\phi\dot\vee\psi\urcorner \leftrightarrow (Tr\ulcorner\phi\urcorner \vee Tr\ulcorner\psi\urcorner)$. However, the addition of $\forall x\forall y(Sent_{Ł_\infty}(x) \wedge Sent_{Ł_\infty}(y) \to (Tr(x\dot\vee y) \leftrightarrow (Tr(x) \vee Tr(y))))$ makes the theory inconsistent.

[15] This question has been raised by Bacon in [2]. While our approach will be semantically-oriented, his approach is proof-theoretic. He analyses what axioms for the conditional can we endorse while avoiding ω-inconsistency.

[16] Actually, some minor additional adjustments need to be made. The other logical expressions are defined non-deterministically too, but in a non-interesting way. For example, negation and disjunction are characterized as follows: $v(\neg\phi) \in \{1 - v(\phi)\}$, and $v(\phi \vee \psi) \in \{max(v(\phi), v(\psi))\}$.

it renders invalid many plausible inferences and principles. So it is interesting to see how much the conditional can be strengthened without making the theory inconsistent or ω-inconsistent[17]. To investigate this issue we now introduce a number of restrictions on the valuations over which the notion of validity will be characterized.

A very odd feature of $\mathcal{NDŁ}^+_\infty$ is that a conditional might not get the value 0 even if its antecedent gets value 1 and its consequent value 0. A way to avoid this is by imposing the following straightforward restriction:

DEFINITION 4.1. *(Semiclassical) A valuation v in a matrix $\mathcal{M}$ is* semiclassical *if and only if for any pair of formulae ϕ_1 and ϕ_2, if $v(\phi_1)$ and $v(\phi_2)$ are both in $\{0,1\}$, then $v(\phi_1 \to \phi_2) = v(\neg\phi_1 \vee \phi_2)$.*

We call this theory $\mathcal{NDŁ}^+_\infty(S)$, for non-deterministic infinitely-valued Łukasiewicz logic with a transparent truth predicate and *semiclassical valuations*. With this new restriction we obtain several inferences that were not validated in $\mathcal{NDŁ}^+_\infty$ (we give examples below).

This is not the only odd feature of $\mathcal{NDŁ}^+_\infty$. Consider two conditionals $\phi_1 \to \phi_2$ and $\phi_3 \to \phi_4$ (where the value of the antecedent is greater than that of the consequent) such that in $\phi_1 \to \phi_2$ the "distance" between ϕ_1 and ϕ_2 is close to 0 and in $\phi_3 \to \phi_4$ it is close to 1. Nothing so far prevents a valuation from assigning to the first a value close to 0 and to the second a value close to 1. To avoid this unpleasant consequence, we can impose the following restriction:

DEFINITION 4.2. *(Uniform$_1$) A valuation v in a matrix $\mathcal{M}$ is* uniform$_1$ *if and only if for any formulae ϕ_1, ϕ_2, ϕ_3 and ϕ_4 such that $v(\phi_1) > v(\phi_2)$ and $v(\phi_3) > v(\phi_4)$, if $v(\phi_1) - v(\phi_2) > v(\phi_3) - v(\phi_4)$, then $v(\phi_1 \to \phi_2) < v(\phi_3 \to \phi_4)$.*

Intuitively this says that if we consider two conditional claims, such that the difference between (the value of) the antecedent and (the value of) the consequent in the first conditional is greater than the difference between (the value of) the antecedent and (the value of) the consequent in the second, the value of the second conditional should be greater than that of the first conditional. For example, if $v(\phi_1) = .8$, $v(\phi_2) = .6$, $v(\phi_3) = .3$ and $v(\phi_4) = .2$, then $v(\phi_1 \to \phi_2) < v(\phi_3 \to \phi_4)$. We call the resulting theory $\mathcal{NDŁ}^+_\infty(U_1)$.

Yet another unsatisfactory feature of $\mathcal{NDŁ}^+_\infty$ is the following. We might have the following two conditionals: $\top \to \lambda$ and $\lambda \to \bot$. Since λ is a Liar

[17] Actually, we have no proof of its ω-consistency, but we strongly suspect that it is in fact ω-consistent. As we will show in section 5, the usual ways to prove ω-inconsistency will not apply.

sentence, its value is $\frac{1}{2}$ in every valuation. Hence, its "distance" from $\top$ is the same as its "distance" from $\bot$. But so far nothing prevents a valuation from assigning very different values (less than 1) to these two formulae. This issue can be dealt with by imposing the following restriction:

DEFINITION 4.3. *(Uniform$_2$) A valuation v in a matrix $\mathcal{M}$ is* uniform$_2$ *if and only if for any formulae ϕ_1, ϕ_2, ϕ_3 and ϕ_4 such that $v(\phi_1) > v(\phi_2)$ and, $v(\phi_3) > v(\phi_4)$, if $v(\phi_1) - v(\phi_2) = v(\phi_3) - v(\phi_4)$, then $v(\phi_1 \to \phi_2) = v(\phi_3 \to \phi_4)$.*

This informally says that if we have two conditional claims, such that the difference between (the value of) the antecedent and (the value of) the consequent is the same in both, the value of the conditionals should be the same. For example, if $v(\phi_1) = .8$, $v(\phi_2) = .6$, $v(\phi_3) = .3$ and $v(\phi_4) = .1$, then $v(\phi_1 \to \phi_2) = v(\phi_3 \to \phi_4)$. We call the resulting theory $\mathcal{NDŁ}^+_\infty(U_2)$.

Now let's introduce one final restriction:

DEFINITION 4.4. *(Bounded from below) A valuation v in a matrix $\mathcal{M}$ is* bounded from below *if and only if for any pair of formulae ϕ_1 and ϕ_2, if $v(\phi_1) > v(\phi_2)$, then $v(\phi_1 \to \phi_2) \geq v(\phi_2)$.*

So if a conditional has a value other than 1, its value has to be greater than the value of its consequent. In other words, an untrue conditional cannot be more untrue than its own consequent. The resulting theory is $\mathcal{NDŁ}^+_\infty(B)$[18].

Naturally, it might be desirable to impose these conditions simultaneously. The strongest theory obtainable is this framework is $\mathcal{NDŁ}^+_\infty(SU_{1,2}B)$, in which all our restrictions play a role[19]. These four restrictions seem fairly natural to us and in fact they all hold in $Ł^+_\infty$. However, we are not claiming that there are no other plausible restrictions that could be imposed without making the conditional fully deterministic[20].

Observe that a valuation for untrue conditionals is acceptable in the theory $\mathcal{NDŁ}^+_\infty(SU_{1,2}B)$ just in case it can be characterized by a strictly

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[18]It has been pointed out to us that this last restriction is too strong for conditionals where the content of the antecedent has nothing to do with the content of the consequent. However, this sort of relevant-oriented worry its out of place in this context. $Ł^+_\infty$'s conditional is not intended as a model of relevant reasoning, and nor are the subtheories of it we are considering.

[19]A similar algebraic approach can be found in [6], chapter 15.

[20]A natural additional restriction that could be imposed is that the valuation functions representing the conditional have to be continuous functions. However, we do not see a strong reason to reject valuation functions that do not fulfill this restriction.

decreasing function[21] f such that $f(x) \geq 1 - x$, where the value of x is given by the difference between the value of the antecedent and the value of the consequent of the conditional.

Let's now define validity for each of the possible theories obtainable by imposing the restrictions above:

DEFINITION 4.5. *(Validity) An argument from the set of formulas Γ to the formula ϕ is* valid$_i$ *($\Gamma \vDash_i \phi$) if and only if every i valuation v in $\mathcal{M}$ that satisfies γ for every $\gamma \in \Gamma$, also satisfies ϕ, where i might be Semiclassical, Uniform$_1$, Uniform$_2$, Bounded from Below or any combination of them.*[22]

In what follows we are going to state a number of facts about these theories. For instance, it is not hard to see that by requiring all valuations to be *Semiclassical*, we obtain:

$$\phi \wedge \neg\psi \vDash_{S+} \neg(\phi \to \psi) \qquad \text{(Negative Material Conditional)}$$

If we demand all valuations to be *Uniform*$_1$, the following become valid:

$$\phi \to \psi \vDash_{U_1^+} (\psi \to \chi) \to (\phi \to \chi) \qquad \text{(Transitivity}_1\text{)}$$
$$\phi \to \psi \vDash_{U_1^+} (\chi \to \phi) \to (\chi \to \psi) \qquad \text{(Transitivity}_2\text{)}$$

It is also straightforward to check that if all valuations are *Uniform*$_2$, then:

$$\vDash_{U_2^+} (\neg\psi \to \neg\phi) \to (\phi \to \psi) \qquad \text{(Contraposition)}$$
$$\vDash_{U_2^+} ((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi)) \qquad \text{(Conj. Intro.)}$$
$$\vDash_{U_2^+} ((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi) \qquad \text{(Disj. Elim.)}$$

Finally, if the valuations are *Bounded from Below*, we get:

$$\vDash_{B+} \phi \to (\psi \to \phi) \qquad \text{(Strong Positive Weakening)}$$
$$(\phi \wedge \psi) \to \chi \vDash_{B+} \phi \to (\psi \to \chi) \qquad \text{(Exportation)}$$

It should be clear that each restriction imposed gives a strictly stronger notion of validity. So the more restrictions we impose on the set of valuations, the more deterministic the conditional is (and the stronger it gets). A rough

[21] We say that a function f is *strictly decreasing* if and only if for all $x_1, x_2 \in dom f, f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

[22] The definition also contemplates the case where Γ is the empty set, so we ambiguously apply 'valid' to both arguments and sentences.

picture of how the restrictions can be put to work is provided in figure 4.1:

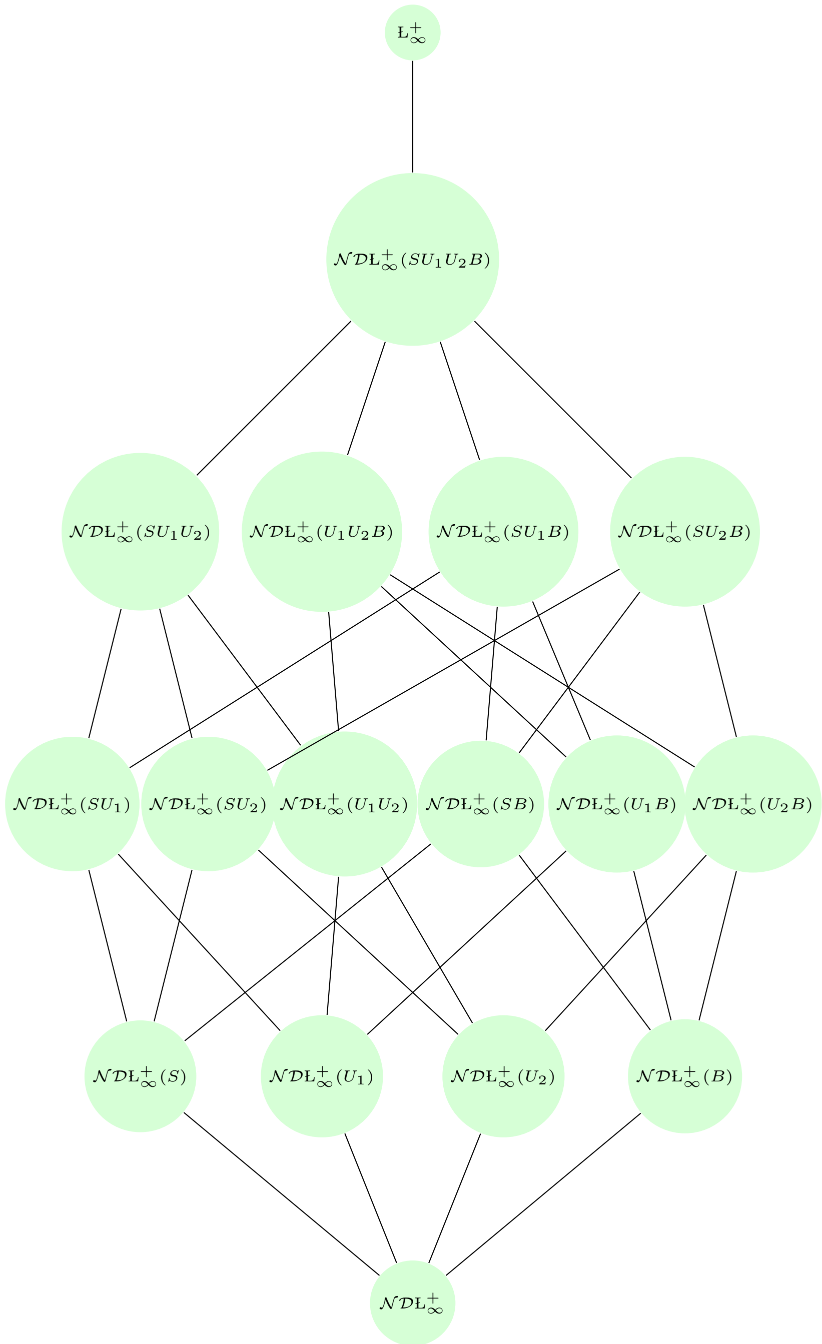

Figure 4.1: All the theories.

A way of spelling out this idea formally is by using the notion of a *refinement*:

DEFINITION 4.6. *(See [1]) A non-deterministic matrix* $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ *is a* refinement *of a non-deterministic matrix* $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ *(notationally,* $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$*) if and only if*

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$,
- $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$,
- $\diamond^{\mathcal{M}_2}(x_1, ..., x_n) \subseteq \diamond^{\mathcal{M}_1}(x_1, ..., x_n)$ *for every n-ary conective* $\diamond$ *and every* $x_1, ..., x_n \in \mathcal{V}_1$.

As we impose more restrictions on the set of valuations, we get refinements of the previous theories. For example, the following obtains:

$\mathcal{NDL}^+_\infty \preccurlyeq \mathcal{NDL}^+_\infty(S) \preccurlyeq \mathcal{NDL}^+_\infty(SU_1) \preccurlyeq \mathcal{NDL}^+_\infty(SU_{1,2}) \preccurlyeq \mathcal{NDL}^+_\infty(SU_{1,2}B) \preccurlyeq$ Ł$^+_\infty$.[23]

In [1], the authors prove that for any pair of non-deterministic matrices $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$, then $\vDash_{M_1} \subseteq \vDash_{\mathcal{M}_2}$. So it immediately follows that:

$\vDash_{\mathcal{ND}Ł^+_\infty} \subseteq \vDash_{S^+} \subseteq \vDash_{SU_1^+} \subseteq \vDash_{SU_{1,2}^+} \subseteq \vDash_{SU_{1,2}B^+} \subseteq \vDash_{Ł^+_\infty}$.

Of course, this is just one example. All the ways in which one of our theories can refine another can be found in Figure 4.1. More specifically, if there is an upward path from a theory $\mathcal{T}_1$ to a theory $\mathcal{T}_2$, then $\mathcal{T}_1 \preccurlyeq \mathcal{T}_2$ and hence $\vDash_{\mathcal{T}_1} \subseteq \vDash_{\mathcal{T}_2}$.

Of particular interest is the fact that the strongest non-deterministic theory we have considered $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$ is such that $\vDash_{SU_{1,2}B^+} \subseteq \vDash_{Ł^+_\infty}$. Moreover, we know that $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$ is a *proper* sublogic of Ł$^+_\infty$, since for instance $\vDash_{Ł^+_\infty} ((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$, but $\nvDash_{SU_{1,2}B^+} ((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$.

Although we will not offer the proof here, the following is a well-known result:

THEOREM 4.7. *(See [7])* Ł$^+_\infty$ *is consistent.*

[23] Although the definition of a refinement is only meant to be applied to non-deterministic matrices, the comparison with Ł$^+_\infty$ is legitimate, since we have shown that every deterministic matrix can be mimicked using some non-deterministic matrix.

Since all subtheories of $Ł^+_\infty$ will be consistent as well, as a corollary of the previous theorem, we can infer:

COROLLARY 4.8. *$\mathcal{NDL}^+_\infty(SU_{1,2}B)$ is consistent.*

Naturally, it also follows that all subtheories of $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$ are consistent.

5. Proving ω-inconsistency?

What about ω-consistency? Restall's [10] proof of $Ł^+_\infty$'s ω-inconsistency rests on the possibility of defining a fusion operator using Łukasiewicz's conditional. But this cannot be done –at least in the same way- in the sort of non-deterministic framework we have presented. Restall defines a fusion operator $^\circ$ in the following way

$\phi^\circ\psi =_{def} \neg(\phi \rightarrow \neg\psi)$

He takes 0 to represent truth and 1 to represent falsity. So 0 is the only designated value and the conditional is defined as restricted substraction: $v(\phi \rightarrow \psi) = v(\psi)\dot{-}v(\phi)$. This means that $v(\phi^\circ\psi) = min(1, v(\phi) + v(\psi))$.

Since we are working with 1 as the only designated value, Łukasiewicz's conditional is defined as follows: $v(\phi \rightarrow \psi) = min(1, 1–v(\phi) + v(\psi))$. This means that $v(\phi^\circ\psi) = 1–min(1, (1–\phi) + (1–\psi))$, which simplifies to

$v(\phi^\circ\psi) = max(0, \psi–(1–\phi)))$

The key aspect of $^\circ$'s behavior is that, for any formula ϕ and any valuation v such that $v(\phi) \neq 1$, there is some finite number n such that the n-fold fusion of ϕ with itself will receive the value 0.

$$v(\underbrace{\phi^\circ(\phi^\circ \ldots (\phi^\circ\phi)...))}_{n-times}) = 0.$$

Why is that so? Because, as we have already stated, $v(\phi^\circ\phi) = max(0, \phi –(1–\phi))$. So $\phi^\circ\phi$ (that is, $\neg(\phi \rightarrow \neg\phi)$) is designed to give a value such that, if $v(\phi) = 1$, then $v(\phi^\circ\phi) = 1$; but if $v(\phi) \neq 1$, then $v(\phi^\circ\phi)$ is strictly less than $v(\phi)$. Moreover, if we fusion $\phi^\circ\phi$ with ϕ, we get a formula whose value is strictly less than $v(\phi^\circ\phi)$; and if we fusion $\phi^\circ(\phi^\circ\phi)$ with ϕ we get a formula whose value is strictly less than $v(\phi^\circ(\phi^\circ\phi))$; and so on until we reach a formula with a value less or equal to $\frac{1}{2}$. In that event, one more fusion with ϕ is enough to reach a formula with value 0.

Using the fusion operator Restall constructs a sequence of sentences S_0, S_1, S_2,... such that S_0 says that not every S_i is true for $i > 0$, and S_{n+1} is the $n+1$-fold fusion of S_0. Then he goes on to show, using a semantic argument, that $Ł_\infty$ is ω-inconsistent, as it declares S_0 true, but also declares, for each i, that S_i is true. The reader is encouraged to see [9] for the details.

Things are not so easy in the non-deterministic framework we have been considering. For example, if we try using the conditional as Restall does it to define a fusion operator, we will not obtain the desired result. To see why, we will consider $\mathcal{NDŁ}^+_\infty$ first. Take any formula ϕ such that $\frac{1}{2} < v(\phi) < 1$. For definiteness, assume that $v(\phi) = .8$. Whereas in $Ł^+_\infty$, $v(\phi \to \neg\phi)$ $=.4$ and hence $v(\neg(\phi \to \neg\phi)) = v(\phi^\circ\phi) = .6$, this formula can receive any undesignated truth-value in $\mathcal{NDŁ}^+_\infty$, which means that its negation can receive any undesignated truth-value as well. Therefore, the fusion of this particular formula with itself, not only does not decrease the value of ϕ, but it might get a higher value than that of ϕ. So $^\circ$, defined in this way, does not work in $\mathcal{NDŁ}^+_\infty$ as it is supposed to. So $\mathcal{NDŁ}^+_\infty$'s presumed ω-inconsistency cannot be proved using this method.

This comes as no surprise, since $\mathcal{NDŁ}^+_\infty$ is a very weak theory. What about the stronger theories? It turns out that the situation is more or less the same. We will consider $\mathcal{NDŁ}^+_\infty(SU_{1,2}B)$, the strongest non-deterministic theory we have presented. Assume once again that $\frac{1}{2} < v(\phi) < 1$ and for definiteness let $v(\phi) = .8$. This time it would be inaccurate to say that $\phi^\circ\phi$ can take any undesignated value. Since valuations are bounded from below, $v(\phi \to \neg\phi) \geq .2$, and hence $v(\phi^\circ\phi) = v(\neg(\phi \to \neg\phi)) \leq .8$[24]. So $v(\phi^\circ\phi)$ is indeed less or equal than $v(\phi)$. In fact, it holds for any formula ϕ and any valuation v that the value of the n-fold fusion of ϕ with itself is less or equal than the value of the $n-1$-fold fusion of ϕ with itself. The problem, of course, is that the restrictions are not enough to guarantee that there is a finite number n (or an infinite number, for that matter) such that the n-fold fusion of ϕ with itself will have value 0. It is perfectly possible for the fusion operation to decrease the values "too slowly", in the sense that the repeated application of this operator produces formulas whose values do not decrease or strictly decrease but with a limit different from 0.

A different way of proving ω-inconsistency is offered by Andrew Bacon in [2]. Bacon shows a proof-theoretic version of the following result:

THEOREM 5.1. *(See [2]) Any transparent truth theory closed under the fol-*

[24] The other conditions seem to be of no help in restricting the valuations further.

lowing rules is ω-inconsistent:[25]
if $\phi \vDash \psi$*, then* $\exists x\phi \vDash \exists x\psi$
$\phi \rightarrow \exists x\psi \vDash \exists x(\phi \rightarrow \psi)$ *(where x is not free in ϕ)*

Proof. See [2] for the proof. Even though his proof is proof-theoretic, it can be mimicked semantically. ■

So if any of the non-deterministic theories we have developed satisfies both these principles, then it is ω-inconsistent. However, we can show that the second rule does not hold in $\mathcal{NDL}^{+}_{\infty}(SU_{1,2}B)$ (and *a fortiori* that it does not hold in any of the weaker theories). Just consider a formula $\psi(x)$ with (at least) x free such that for no x-variant v' of v it holds that $v'(\phi) \leq v'(\psi)$ but $sup\{v'(\psi) : v' \text{ is an } x\text{-variant of } v\} = v(\phi)$. In Ł$^{+}_{\infty}$, this valuation will make both the premise and the conclusion true. However, in $\mathcal{NDL}^{+}_{\infty}(SU_{1,2}B)$ or in any of the weaker non-deterministic theories, there is no guarantee that the conclusion is true.

There might be some other way to define fusion, or to prove ω-inconsistency, but currently we are not aware of any. Now, if ω-inconsistency cannot be proved for these theories, it should be possible to construct nice models for them. However, it is not at all obvious to us at the moment how to do this. Clearly, we cannot show that a nice model exists by defining a monotone jump operator on the interpretations of the truth predicate, as is usually done in a number of many-valued theories. The reason is that $\mathcal{NDL}^{+}_{\infty}(SU_{1,2}B)$'s conditional (just as Ł$^{+}_{\infty}$'s conditional) is not a monotone operation on the set of values.

Another strategy would be to use Brower's fixed point theorem, according to which every continuous function on the set of k-tuples of real numbers in the interval $[0,1]$ has a fixed point. In fact, in [6] Field uses this theorem to show that the propositional fragment of Ł$^{+}_{\infty}$ has a nice model. However, his proof applies only to the propositional part of the language and it depends on the conditional being a continuous function, something which holds in Ł$^{+}_{\infty}$ but does not hold in our theories[26].

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[25] Actually, Bacon makes a distinction between strongly ω-inconsistent theories and weakly ω-inconsistent theories. However, for our purposes this distinction will be unnecessary.

[26] Perhaps it is possible to modify the conditional so that it is a continuous function (or more precisely, a continuous function relative to each valuation). But even in that case it is still unclear to us whether this strategy could be applied to the the full quantificational language.

6. Determinate truth in non-deterministic semantics

One of the known problems of certain paracomplete theories is that they are too weak to express the idea that certain sentences are not determinately true. An additional virtue of $Ł^+_\infty$ is that it overcomes this defect. Its conditional can be used to define a nice determinately operator:

$$D\phi =_{df} \neg(\phi \rightarrow \neg\phi).$$

The reader will notice that $D\phi$ is the same as $\phi^\circ\phi$. So in $Ł^+_\infty$ the determinately operator is just a case of the fusion operator.

Now, in the previous section we have seen that the fusion operator cannot be used -at least not in the standard way- to prove the ω-inconsistency of the non-deterministic theories. Since the determinately operator is just a limit case of the fusion operator, a natural worry is that these theories loose one very attractive feature of $Ł^+_\infty$, namely, the ability to consistently add such an operator, and *a fortiori*, the ability to express the idea that certain sentences are not determinately true.

However, we will show that there is no reason for concern. Following [6], we will say that we should expect the following from a nice determinately operator:

1. If $v(\phi) = 1$, then $v(D\phi) = 1$.
2. If $v(\phi) \leq v(\neg\phi)$, then $v(D\phi) = 0$.
3. If $0 < v(\phi) < 1$, then $v(D\phi) \leq v(\phi)$[27].
4. If $v(\phi) \leq v(\psi)$, then $v(D\phi) \leq v(D\psi)$.

Fortunately, we can show that all these conditions hold for the operator D in $\mathcal{ND}Ł^+_\infty(SU_{1,2}B)$, the strongest of the theories we have been considering. More precisely,

[27] In [6], p. 235-36, Field flirts with the stronger:
If $0 < v(\phi) < 1$, then $v(D\phi) < v(\phi)$,
which will not hold in our non-deterministic theories, but ends up using 3. For the stronger version to hold we would need to strengthen *bounded from below* by imposing that $v(\phi \rightarrow \psi) > v(\psi)$. As an anonymous referee suggests, it is this what is responsible for $Ł^+_\infty$'s ω-inconsistency, because it allows us to define an operator $D^*\phi$ expressing the idea that ϕ is determinate at all countable ordinals. This in turn would be enough to define classical negation, since $v(D^*\phi) = 1$ whenever $v(\phi) = 1$, and $v(D^*\phi) = 0$, otherwise. However, as far as we can see, this condition is not enough to define classical negation in our non-deterministic theories, because if $v(\phi) \in (\frac{1}{2}, 1)$, we can always pick a value r in $[0, v(\phi))$ for $D^*\phi$. So if λ_* is the sentence $\neg D^*\lambda_*$, there might be valuations v such that $v(\lambda_*) = v(\neg D^*\lambda_*) \in (\frac{1}{2}, 1)$.

THEOREM 6.1 (A determinately operator). *$\mathcal{NDL}^+_\infty(SU_{1,2}B)$ contains a nice determinately operator D.*

PROOF SKETCH. Just as before, let $D\phi$ be $\neg(\phi \to \neg\phi)$. It is clear that every valuation v satisfies condition 2. above. In addition, if v is *semiclassical*, then condition 1. holds, if v is *bounded from below*, then condition 3. holds, and if v is both $uniform_1$ and $uniform_2$, then condition 4. holds. ■

These properties are not only nice in themselves. Once D is shown to satisfy them, we can prove that D is non-idempotent, in the sense that sometimes $v(D\phi) \neq v(DD\phi)$. This is specially important in the case of Liar-like sentences λ_n such that $v(\lambda_n) = v(\neg D^n Tr\ulcorner\lambda_n\urcorner)$, where D^n stands for n iterations of the D operator. We not only have $v(D\lambda) = 0$, but for any λ_n such that λ_n is $\neg D^n Tr\ulcorner\lambda_n\urcorner$, we can prove $v(D^{n+1}\lambda_n) = 0$. In other words, for any Liar sentence expressible in the language we can say in the language that there is a sense in which it is not determinately true[28].

7. Conclusion

We will finish by mentioning a similar approach explored recently by Bacon in [2]. Bacon considers two theories: BCKN and BCKD. BCKN has models where the space of values is not linearly ordered, so this feature can be exploited to show that $\phi \to \exists x\psi \nvDash \exists x(\phi \to \psi)$. But since we are not considering non-linearly ordered matrices in this paper, we will ignore it. In the case of BCKD, the presence of the Connectivity axiom guarantees that the theory is true only on linearly ordered spaces of values. In particular, Bacon considers the following semantic definition for BCKD's conditional:

$$v(\phi \to \psi) = \begin{cases} 1 & \text{if } v(\phi) \leq v(\psi) \\ v(\psi) & \text{otherwise} \end{cases}$$

Then he goes on to show that the inference from $\phi \to \exists x\psi$ to $\exists x(\phi \to \psi)$ fails for BCKD's conditional. However, this inference fails only at the cost of using models that are not compatible with a transparent truth predicate. Consider again a Curry sentence δ such that δ is $\mathrm{T}(\delta) \to \bot$. It is straightforward to see that the above definition for the conditional cannot consistently assign a truth value to δ.

In the case of the non-deterministic theories presented above, there is no analogous problem. Curry's Paradox is blocked (every Curry sentence

[28]The reader can see [6] for how to generalize this result into the transfinite.

receives a stable non-designated value) and hence triviality is avoided. But are these theories ω-consistent? We can conjecture that that they are but, certainly, the need for an ω-consistency proof remains[29].

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Acknowledgements. Earlier versions of this material were presented at conferences in Buenos Aires and Munich. We would like to thank the audiences of those conferences for their valuable comments and suggestions. We are particularly grateful to Eduardo Barrio, Natalia Buacar, Catrin Campbell-Moore, Walter Carnielli, Eleonora Cresto, Alfonso Losada, Julien Murzi, Ignacio Ojea, Lavinia Picollo, Thomas Schindler, Johannes Stern, Damián Szmuc, Diego Tajer, Paula Teijeiro, Peter Verdée and two anony-

[29] Another open problem is whether the non-deterministic theories have a complete axiomatization. As an anonymous referee pointed out to us, if this were the case, we would have a strong argument in favor of these theories. As far as we can see, the issue is not at all trivial. For example, it is quite easy to show that $Ł^{+}_{\infty}$ is not compact and that, as a consequence, it is not axiomatizable (a proof of this claim can be found in [8], p. 240, exercises 8-9). However, the most direct proof of this fact depends on the possibility of using the conditional to define a *fission* operator -dual to the fusion operator- something that cannot be done in the usual way with the non-deterministic conditionals.

mous reviewers of *Studia Logica*. We would like to add that the paper could not have been written without the financial aid of CONICET.

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