J Philos Logic
DOI 10.1007/s10992-014-9342-6

The Logics of Strict-Tolerant Logic

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Received: 14 December 2013 / Accepted: 5 December 2014

Abstract Adding a transparent truth predicate to a language completely governed by classical logic is not possible. The trouble, as is well-known, comes from paradoxes such as the Liar and Curry. Recently, Cobreros, Egré, Ripley and van Rooij have put forward an approach based on a non-transitive notion of consequence which is suitable to deal with semantic paradoxes while having a transparent truth predicate together with classical logic. Nevertheless, there are some interesting issues concerning the set of metainferences validated by this logic. In this paper, we show that this logic, once it is adequately understood, is weaker than classical logic. Moreover, the logic is in a way similar to the paraconsistent logic LP.

Keywords Transparent truth · Strict-tolerant logic · Semantic paradoxes · Transitivity · Substructural logic

1 Introduction

Adding a transparent truth predicate to a language completely governed by classical logic is not possible. The trouble, as is well-known, comes from paradoxes such as

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Published online: 31 December 2014

the Liar and Curry. Because of these paradoxes, theories of truth typically use a non-transparent truth predicate to preserve classical reasoning. Revisionary approaches to logic take a different path. They deal with semantic paradoxes by weakening classical logic. However, Curry's paradox makes this task really complicated. Any (non-substructural) logic featuring a conditional connective validating Modus Ponens and other plausible rules together with a transparent truth predicate has to face Curry's paradox.

Non-classical approaches that contain a transparent truth predicate are usually based in some way on Kripke's work [13]. For example, theories of this kind can be found in [19], [4] and [10]. Kripke's construction starts from a classical model for a first-order base language $\mathcal{L}$[1] without any truth predicate, and provides a way to generate a model for the language $\mathcal{L}^+$ that adds a transparent truth predicate $T(x)$ to $\mathcal{L}$. Kripke uses three-valued models for $\mathcal{L}^+$ with the set $\{1, \frac{1}{2}, 0\}$ of values, and employs Kleene's strong valuation schema. According to this schema, negation $\neg$ is defined as 1 minus the value of the negated formula; conjunction $\wedge$ is defined as the minimum of the values of the conjuncts, and universal quantification $\forall$ as the minimum of the values over all assignments that differ at most on the object they assign to the variable bounded by the quantifier. It is easy to define disjunction $\vee$, material conditional $\supset$, material biconditional $\equiv$ and the existential quantifier $\exists$ using the other logical expressions.

We will assume that the language in question has enough resources to talk about itself. For this purpose it seems convenient to use Ripley's method in [22]. The idea is to divide the individual constants into two sets: the *ordinary names*, which work as the usual individual constants, receiving their denotation from each model, and the *distinguished names*, which get their denotations independently. Basically, they get interpreted by a one-to-one function τ from distinguished names onto formulas of $\mathcal{L}^+$ such that in each model, the denotation of a distinguished name n is $\tau(n)$. This is enough to get the usual self-referential paradoxical sentences. For instance, given a distinguished name n_1, we can let $\tau(n_1)$ be $\neg T n_1$, which is a Liar sentence. Following the usual practice, for any sentence A in $\mathcal{L}^+$, we use the notation $\langle A \rangle$ for the name of A.[2]

Now, it is not complicated to produce different strong Kleene models, with the additional feature that the value assigned to a sentence of the form $T\langle A \rangle$ is always the same as the value assigned to A itself, for every sentence A. These models are usually called transparent.

It is well-known that the notion of logical consequence can be defined in different ways in this setting. Usually, logical consequence is understood as absence of

[1]It will be useful to assume that $\mathcal{L}$ contains a falsity constant $\bot$ and a truth constant $\top$, in addition to the usual connectives and quantifiers.

[2]To simplify things, our language lacks an identity predicate. Notice that because of the way in which we handle self-reference (denotations are fixed through the models), identity is not really needed for that purpose. In any case, as Ripley himself points out, a different approach could be used -such as employing Peano arithmetic as our naming system or having a quote-name forming device in the language. The results we reach below do not depend in any way on this issue, although the proofs might have been slightly different.

a countermodel. Classically, a countermodel of an argument from a set of formulas Γ to a set of formulas Δ is a model that assigns 1 to every member of Γ and 0 to every member of Δ.[3] Nevertheless, there are multiple ways to tweak this notion in a three-valued framework. Some of these ways result in relatively familiar logics: one strategy, resulting in the logic K_3^+ with transparent truth, is to take a countermodel to be an interpretation that assigns 1 to every member of Γ and some value less than 1 to every member of Δ. Another way, resulting in the logic LP^+ with transparent truth, is to take a countermodel to be an interpretation that assigns some value greater than 0 to every member of Γ and 0 to every member of Δ.[4] The problems with this kind of views are well-known. The material conditional exhibits an odd behavior in these logics: LP^+ does not validate Modus Ponens (A , $A \supset B \nvDash_{LP^+} B$), while K_3^+ does not validate Identity ($\nvDash_{K_3^+} A \supset A$). Both Field [10] and Beall [4] have worked on supplementing such theories with a suitable conditional, one that both detaches, validates Identity and validates all T-biconditionals. But Curry's paradox makes this task quite complicated: any (non-substructural) theory with a transparent truth predicate cannot have a conditional connective validating, for example, Modus Ponens and Contraction, or Modus Ponens and Conditional Proof. As a result, neither Beall's nor Field's conditional seem entirely satisfactory.[5]

2 The Logic ST^+

The non-transitive approach we will consider provides a different way of using Kripke models to define a logic of truth. It characterizes validity in a way that makes the introduction of an extra conditional unnecessary. The key is to use different standards for the premises and the conclusions in the definition of validity. Roughly, a set of formulas Δ is a consequence of a set of formulas Γ if whenever all the premises in Γ hold up to some standard, the members of Δ hold to some other standard.

In particular, to deal with semantic paradoxes (and also with the Sorites paradox), it is useful to focus on the three-valued strict-to-tolerant consequence relation $\vDash_{ST^+}$ (see [6–9, 21] and [22]). Namely, strict-to-tolerant consequence goes from a strictly true set of premises (that is, every premise takes the value 1) to a tolerantly true set of conclusions (that is, at least one conclusion does not take value 0). More formally:

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[3] We will work in a multiple conclusions setting. Nothing important hangs on this, but the approach we are going to consider below makes use of multiple conclusions, so working with them from the start will simplify the discussion.

[4] We use the notation L^+ to stress that the logic L already contains a transparent truth predicate, that is, a predicate $T(x)$ such that for any formula A and any valuation v, $v(T\langle A\rangle) = v(A)$.

[5] There are also additional problems with these theories. For instance, the weakness of the conditional produces complications with restricted quantifications (see [4]). Also, these theories cannot include a naive notion of validity because of the Validity Paradox (see [5]). However, we will not address these issues here.

Definition 1 (Strict-Tolerant consequence) We say that Δ is a strict-tolerant consequence of Γ (i.e. $\Gamma \vDash_{ST^+} \Delta$) if and only if for every valuation v, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$, then $v(\delta) > 0$ for some $\delta \in \Delta$.

Hence, a countermodel of an argument from a set of premises Γ to a set of conclusions Δ is a model that assigns 1 to every member of Γ and 0 to every member of Δ (just like a classical countermodel!).

Since we want to discuss a transparent truth predicate, the models of ST^+, just like those of K_3^+ and LP^+, respect the transparency requirement:

For every valuation v, $v(T\langle A\rangle) = v(A)$

A different way to compare ST^+ with K_3^+ and LP^+ is as follows: in many-valued logics, falsity and non-truth do not coincide. In K_3^+ validity is based on strict-truth preservation (or true-only preservation) and in LP^+ on tolerant-truth preservation (or non-falsity preservation). K_3^+ and LP^+ are different logics, because both systems have different designated values. The duality between K_3^+ and LP^+ is such that the relative merits one logic may claim over the other can usually be turned into relative limitations. In ST^+ validity is no longer defined as the preservation of some designated truth-value from premises to conclusions, but rather as a weakening of standards when going from premises to conclusions.

It is easy to see that ST^+ is, in a sense, stronger than both K_3^+ and LP^+: any ST^+-countermodel is automatically both a K_3^+- and an LP^+-countermodel, but there are K_3^+- and LP^+-countermodels that are not ST^+-countermodels. According to [9], p.841:

> the key advantage of our approach, from which a number of other advantages will follow, lies in its keeping to classical logic.

In fact considering its T-free fragment, ST^+ is exactly classical logic augmented with quote-names. An argument from a set of premises Γ to a set of conclusions Δ is ST^+-valid if it is CL-valid (where CL stands for classical logic). So ST^+ captures classical validity. Moreover, ST^+ conservatively extends CL: the only differences appear with arguments that involve the truth predicate.

Theorem 1 (Conservativeness, [22]) *For any* Γ, $\Delta \subseteq \mathcal{L}$, $\Gamma \vDash_{ST} \Delta$ *iff* $\Gamma \vDash_{ST^+} \Delta$.

For the extended language, things change though. For example, $\lambda \wedge \neg\lambda$ (where λ is a Liar sentence) is a valid sentence of ST^+, whereas in classical logic no contradiction can be valid.

Moreover, the following stronger result holds in this approach:

Theorem 2 (Uniform substitution on $\mathcal{L}$, [22]) *If* $\Gamma \vDash_{ST} \Delta$ *and* $\Gamma \cup \Delta \subseteq \mathcal{L}$, *then* $\Gamma^* \vDash_{ST^+} \Delta^*$, *where* $*$ *is any uniform substitution (of open formulas for predicates, avoiding bound variable conflict in the usual ways) on* $\mathcal{L}^+$.

This means that Excluded Middle, Explosion, *Modus Ponens*, and other classically valid principles that have been given up by other approaches in the face of paradox are valid in this framework. The first result ensures that these continue to hold where $T(x)$ is not involved, and the latter ensures that they extend to all cases where $T(x)$ is involved.

Nevertheless, in order to avoid triviality, ST^+ has to lose some traditional properties concerning what we will call 'metainferences'. Metainferences are principles *about* the consequence relation. It is important to be clear on the difference between a valid argument and a metainference. A valid argument establishes a certain relation between sets of formulas. A metainference is a closure property on the set of valid arguments (see [22], p. 354). Clear cases of metainferences are the following, where A, B and C are schematic formulas:

– if $A \vDash B$, then $\vDash A \supset B$ (Conditional Proof)
– if $A \vDash B$ and $B \vDash C$, then $A \vDash C$ (Transitivity)
– if $A \vDash B$ and $C \vDash B$, then $A \vee C \vDash B$ (Proof by Cases)

Not all metainferences usually associated with classical logic will hold in ST^+. First among these is transitivity. There are some wffs A, B and C such that $A \vDash_{ST^+} B$ and $B \vDash_{ST^+} C$, but $A \nvDash_{ST^+} C$. For example, consider a Liar sentence λ again. This sentence has to take value $\frac{1}{2}$ on every ST^+ model. Since ST^+ requires countermodels to go from 1 to 0, there is no ST^+-countermodel of the argument from p to λ (where p is any atomic formula); thus, $p \vDash_{ST^+} \lambda$. In the same way, there is no ST^+-countermodel of the argument from λ to q (where q is another atomic formula); hence $\lambda \vDash_{ST^+} q$. However, it is easy to find an ST^+-countermodel of the argument from p to q; just assign 1 to p and 0 to q. Therefore, $p \nvDash_{ST^+} q$. So, $\vDash_{ST^+}$ is not transitive. To be fair, let us note that it is precisely this sort of failures that allow ST^+ to avoid the Liar paradox and Curry's paradox, among other things.

The failure of transitivity implies -given the Deduction Theorem, which holds for ST^+- that the following version of *Modus Ponens* fails as well. Suppose we have valid sentences of the form A and $A \supset B$. It does not follow that B is also a valid sentence. An easy way to see this is by taking A to be λ and B to be $\bot$.

A version of *Modus Tollens* fails too: even if $\neg B$ and $A \supset B$ were valid sentences, it does not follow that $\neg A$ is valid as well (take B as λ and A as $\top$). Furthermore, Explosion, Disjunctive Syllogism and certain versions of *Reductio* do not hold as metainferences either.

Observe that all these metainferences are invalid LP^+ inferences. So there seems to be a similarity between LP^+'s inferences and ST^+'s metainferences. However, whereas LP^+'s notion of validity is clear enough, the non-standard features of ST^+'s concept of validity make it hard to grasp exactly what this logic amounts to. Below we will argue that the correct metainferences of ST^+ can be understood as the valid inferences of LP^+.

There is another interesting notion which is related to the concept of a metainference. Informally, we say that an argument is *externally valid relative to a theory* $\mathcal{T}$ if the result of adding the premises of the argument as axioms to $\mathcal{T}$ is enough to prove as theorems the conclusions of the argument (we will provide a more rigorous definition below). For classical logic there is no difference between validity and external validity. Every valid argument is externally valid and viceversa. It turns out that for substructural logics such as ST^+ these two notions no longer coincide. Although every classically valid argument is ST^+-valid, it is not the case that every classically valid argument is externally valid according to ST^+. Moreover, we will prove that the set of externally valid ST^+-arguments is exactly the set of LP^+-valid arguments.

We will claim that these facts can be used to argue that ST^+ is similar to LP^+ and, moreover, that the acknowledged weaknesses of LP^+ are in a way weaknesses of ST^+ too.

The next three sections of the paper are structured as follows. In the next one we prove that the set of valid ST^+-metainferences is just the set of valid LP^+-inferences. So, there is a sense in which ST^+'s notion of consequence is exactly LP^+. In Section 4 we introduce the notion of external logic and prove, using a three-sided sequent calculus, that the external logic of ST^+ is just the logic of LP^+. Section 5 contains some additional conceptual considerations against ST^+ that follow from the results of Sections 3 and 4.

3 LP^+ and ST^+'s Metainferences

In this section, we will prove that there is a way in which the 'logic' of the metainferences of ST^+ is just the logic LP^+. As expected, the proof we will offer in this section uses some new notions.

Definition 2 (*Schematic formula*) A *schematic formula* is a propositional formula that, instead of propositional letters $p, q, ...$ has schematic letters $A, B, C....$

For example, $A \wedge B$, A and $A \supset (B \supset C)$ are schematic formulas.

Definition 3 (*Replacement function*) A *replacement function* is a function σ from schematic formulas to formulas of $\mathcal{L}^+$ such that:

$\sigma(\phi)$[6] is an arbitrary formula of $\mathcal{L}^+$, when ϕ is an atomic schematic formula.

$\sigma(\phi) = \sigma(\psi) \wedge \sigma(\xi)$ when ϕ is a schematic formula $\psi \wedge \xi$.

$\sigma(\phi) = \sigma(\psi) \vee \sigma(\xi)$ when ϕ is a schematic formula $\psi \vee \xi$.

$\sigma(\phi) = \neg\sigma(\psi)$ when ϕ is a schematic formula $\neg\psi$.

Sometimes we will use the terminology $\sigma(\Delta)$, when Δ is a set of sentences. The intended meaning is:

$\sigma(\Delta) = \{\sigma(\delta) | \delta \in \Delta\}$

Definition 4 (*Metainference*) We will say that a *metainference* is a sentence of the following kind, where every formula ψ_i or $\delta \in \Delta_i$ is schematic:

[6] In the proof we use Greek letters to represent meta-schematic variables. We need meta-schematic letters because the proof is about schematic formulas. For example, ϕ may represent $A \vee B$; and $A \vee B$ may represent $(p \wedge q) \vee (r \wedge s)$ under some replacement function.

(For every uniform replacement of ψ_i and $\delta \in \Delta_i$ by formulas of the language) If $\sigma(\Delta_1) \vDash \sigma(\psi_1)$ and ... and $\sigma(\Delta_n) \vDash \sigma(\psi_k)$, then $\sigma(\Delta) \vDash \sigma(\psi)$.[7]

This general meta-schema is meant to include all the examples we gave before. For example, Conditional Proof becomes: (For every uniform replacement σ of A, B by formulas of the language) if $\sigma(A) \vDash \sigma(B)$, then $\vDash \sigma(A \supset B)$.

It could be useful to clarify what a (schematic) metainference is *not*. For instance, the claim 'if $p \vDash q$, then $p \vDash p \vee q$', is not a metainference but an instance of a metainference, because p and q are formulas belonging to the object language. A metainference is a general claim to the effect that if certain *kinds* of inferences hold, then another *kind* of inference holds as well. This explains why we use schematic formulas.

We will argue that every ST^+ schematic metainference can be translated into an LP^+ schematic inference using a conditional instead of a consequence symbol. More precisely, the idea is the following (where t is a translation function):

Definition 5 If $\Delta \vDash \psi$ is an ST^+ inference schema, its LP^+ *translation* $t(\Delta \vDash \psi)$ is defined as follows:
$t(\Delta \vDash \psi) = \bigwedge \Delta \supset \psi$, and
$t(\vDash \psi) = \psi$.

The result we are after can be stated as follows:

Theorem 3 (First Collapse Result) *The following two sentences are equivalent:*
(a) (For every uniform replacement σ of $\delta \in \Delta_i$ and ψ_i by formulas of the language) if $\sigma(\Delta_1) \vDash_{ST^+} \sigma(\psi_1)$ and ... and $\sigma(\Delta_n) \vDash_{ST^+} \sigma(\psi_n)$, then $\sigma(\Delta) \vDash_{ST^+} \sigma(\psi)$.
(b) (For every uniform replacement σ of $\delta \in \Delta_i$ and ψ_i by formulas of the language), $\sigma(t(\Delta_1 \vDash \psi_1))$, ..., $\sigma(t(\Delta_n \vDash \psi_n)) \vDash_{LP^+} \sigma(t(\Delta \vDash \psi))$.[8,9]

[7] To keep things as simple as possible, in this section we will not use multiple conclusions, and we will assume that the sets Δ_i are always finite. Nothing important hangs on this though.

[8] Notice that we need to quantify over uniform replacements in both sentences. If the quantifiers were to range over the entire equivalence, the statement would be false. In other words, it is *false* that:
(For every uniform replacement σ of $\delta \in \Delta_i$ and ψ_i by formulas of the language), the following sentences are equivalent:
(a) if $\sigma(\Delta_1) \vDash_{ST^+} \sigma(\psi_1)$ and ... and $\sigma(\Delta_n) \vDash_{ST^+} \sigma(\psi_n)$, then $\sigma(\Delta) \vDash_{ST^+} \sigma(\psi)$.
(b) $\sigma(t(\Delta_1 \vDash \psi_1))$, ..., $\sigma(t(\Delta_n \vDash \psi_n)) \vDash_{LP^+} \sigma(t(\Delta \vDash \psi))$.
Just as a counterexample, take
(a) $A, A \supset B \vDash_{LP^+} B$, and
(b) if $\vDash_{ST^+} A$ and $A \vDash_{ST^+} B$, then $\vDash_{ST^+} B$.
Now replace A by p and B by q. So the first claim, '$p, p \supset q \vDash_{LP^+} q$', would be false, while the second claim, 'if $\vDash_{ST^+} p$ and $p \vDash_{ST^+} q$, then $\vDash_{ST^+} q$', would be true, because its antecedent is false (p is not a tautology in ST^+).

[9] An anonymous referee has suggested the following counterexample:
(a) It is true that (for every replacement): if $A \vDash_{ST^+} B \wedge \neg B$, then $A \vDash_{ST^+} C$.
(b) But it is false that (for every replacement): $A \supset (B \wedge \neg B) \vDash_{LP^+} A \supset C$.
However, this is not a counterexample to our theorem. Because while observation (b) is correct, observation (a) is incorrect. In particular, $p \vDash_{ST^+} \lambda \wedge \neg\lambda$, but $p \nvDash_{ST^+} q$.

We will provide a proof of the (b) to (a) direction first. For that we need to prove the following lemma:

Lemma 1 *For every replacement function* σ, *if* $\sigma(\Delta_i) \vDash_{ST^+} \sigma(\psi_i)$, *then* $\vDash_{LP^+} \sigma(\bigwedge \Delta_i \supset \psi_i)$ *(in other words,* $\vDash_{LP^+} \sigma(t(\Delta_i \vDash \psi_i))$*).*

Proof Since Conditional Proof holds in ST^+, if for every σ, $\sigma(\Delta_i) \vDash_{ST^+} \sigma(\psi_i)$, then of course for every σ, $\vDash_{ST^+} \sigma(\bigwedge \Delta_i \supset \psi_i)$. And given that every valid ST^+-sentence is a valid LP^+-sentence, we can obtain $\vDash_{LP^+} \sigma(\bigwedge \Delta_i \supset \psi_i)$. □

Now we can proceed to prove the first part of Theorem 3.

Proof (From (b) to (a)) Assume that $\sigma(\Delta_1) \vDash_{ST^+} \sigma(\psi_1)$ and ... and $\sigma(\Delta_n) \vDash_{ST^+} \sigma(\psi_n)$. By Lemma 1, we obtain $\vDash_{LP^+} \sigma(t(\Delta_1 \vDash \psi_1) \wedge ... \wedge t(\Delta_n \vDash \psi_n))$. Therefore, given (b), $\vDash_{LP^+} \sigma(t(\Delta \vDash \psi))$. This step is justified because LP^+ has a transitive consequence relation (unlike ST^+).This means that $\vDash_{LP^+} \sigma(\bigwedge \Delta \supset \psi)$. Since every LP^+-valid sentence is an ST^+-valid sentence, $\vDash_{ST^+} \sigma(\bigwedge \Delta \supset \psi)$. By the Deduction Theorem, which holds in ST^+, we can obtain $\sigma(\Delta) \vDash_{ST^+} \sigma(\psi)$. □

The (a) to (b) direction is a bit more complicated. We will need to prove a few lemmata in the way.

Proof (From (a) to (b)) Let σ be a replacement of the schematic formulas $\delta \in \Delta$ and ψ_i by formulas of the language $\mathcal{L}^+$ and v a valuation such that:
$v(\sigma(t(\Delta_1 \vDash \psi_1)))$= 1 or $\frac{1}{2}$, ..., $v(\sigma(t(\Delta_n \vDash \phi_n)))$=1 or $\frac{1}{2}$, and
$v(\sigma(t(\Delta \vDash \psi)))$=0.

We want to prove that for some uniform replacement σ' of the schematic formulas $\delta \in \Delta_i$ and ψ_i, the following obtains: $\sigma'(\Delta_1) \vDash_{ST^+} \sigma'(\psi_1)$ and ... and $\sigma'(\Delta_n) \vDash_{ST^+} \sigma'(\psi_n)$, but $\sigma'(\Delta) \nvDash_{ST^+} \sigma'(\psi)$.

Now we should define a function r that will help us to get the needed formulas from the valuation v. The composed function $r \circ \sigma$ will be the function σ' we are looking for.

Definition 6 The function r goes from formulas P of the object language $\mathcal{L}^+$ to formulas of $\mathcal{L}^+$ and is such that:

– When P has the form $Fa_1...a_n$ or $T(\langle Q \rangle)$ or $\forall x Q$, then

  – $r_v(P) = \bot$ when v(P) = 0.
  – $r_v(P) = \top$ when v(P) = 1.
  – $r_v(P) = \lambda$ when v(P) = $\frac{1}{2}$.

- $r_v(P \vee Q) = r_v(P) \vee r(Q)$.
- $r_v(P \wedge Q) = r_v(P) \wedge r(Q)$.
- $r_v(\neg P) = \neg r_v(P)$.

Now we can prove the following lemma:

Lemma 2 *If $v(P) = \frac{1}{2}$, then $r_v(P) \equiv \lambda$. If $v(P) = 1$, then $r_v(P) \equiv \top$. If $v(P) = 0$, then $r_v(P) \equiv \bot$.*

Proof By induction over complexity. If P has the form $Fa_1...a_n$ o $T\langle A\rangle$ or $\forall x A$, then it is trivial.

If P has the form $Q \wedge R$, then:

- Suppose that $v(Q \wedge R) = 1$. Then $v(Q) = v(R) = 1$. Therefore by the inductive hypothesis $r_v(Q) \equiv r_v(R) \equiv \top$, so $r_v(Q \wedge R) = r_v(Q) \wedge r_v(R) \equiv \top$.
- Suppose that $v(Q \wedge R) = 0$. Then $v(Q) = 0$ or $v(R) = 0$. Suppose without loss of generality that $v(Q) = 0$. By the inductive hypothesis, $r_v(Q) \equiv \bot$, so $r_v(Q \wedge R) = r_v(Q) \wedge r_v(R) \equiv \bot$.
- Suppose that $v(Q \wedge R) = \frac{1}{2}$. Then $v(Q) = v(R) = \frac{1}{2}$, or $v(Q) = \frac{1}{2}$ and $v(R) = 1$, or $v(Q) = 1$ and $v(R) \equiv \frac{1}{2}$. In the first case, it follows that $r_v(Q) \equiv \lambda \equiv r_v(R)$, so $r_v(Q \wedge R) \equiv \lambda$. In the second case (without loss of generality), it follows that $r_v(Q) \equiv \lambda$ and $r_v(R) \equiv \top$. So $r_v(Q \wedge R) = r_v(Q) \wedge r_v(R) \equiv (\lambda \wedge \top) \equiv \lambda$.

The remaining cases are similar. □

Now we can show that:

Lemma 3 *For every replacement function σ, if $v(\sigma((\bigwedge \Delta_i) \supset \psi_i)) = 0$, then $r_v(\sigma(\Delta_i)) \nvDash_{ST^+} r_v(\sigma(\psi_i))$.*

Proof Assume that $v(\sigma(\bigwedge \Delta_i \supset \psi_i)) = 0$. Then for all $\delta \in \Delta_i$, $v(\sigma(\delta)) = 1$, and $v(\sigma(\psi_i)) = 0$. By Lemma 2, this implies that $r(\sigma(\Delta_i)) \equiv \top$, while $r(\sigma(\psi)) \equiv \bot$. Therefore, $r_v(\sigma(\Delta)) \nvDash_{ST^+} r_v(\sigma(\psi))$, since all valuations will make the premises true and the conclusion false. □

On the other hand, we can prove:

Lemma 4 *If $v(\sigma(\Delta_i \supset \psi_i)) = 1$ or $\frac{1}{2}$, then $r(\sigma(\Delta_i)) \vDash_{ST^+} r_v(\sigma(\psi_i))$.*

Proof CASE 1. Assume that $v(\sigma(\Delta_i \supset \psi_i))$= 1. Then $v(\sigma(\psi_i)) = 1$ or $v(\sigma(\bigwedge \Delta_i))$= 0. If $v(\sigma(\psi_i)) = 1$, then by Lemma 2, $r_v(\sigma(\psi_i)) \equiv \top$. Therefore $r_v(\sigma(\Delta_i)) \vDash_{ST^+} r_v(\sigma(\psi_i))$. If $v(\sigma(\Delta_i))$= 0, then for some $\delta \in \Delta_i$, $v(\sigma(\delta)) = 0$. So by Lemma 2, $r_v(\sigma(\delta)) \equiv \bot$. Therefore $r_v(\sigma(\Delta_i)) \vDash_{ST^+} r_v(\sigma(\psi_i))$.

CASE 2. Assume that $v(\sigma(\bigwedge \Delta_i \supset \psi_i))$= $\frac{1}{2}$. Then:

2A. $v(\sigma(\bigwedge \Delta_i)) = \frac{1}{2}$ and $v(\sigma(\psi_i)) = \frac{1}{2}$. By Lemma 2, this means that $r_v(\sigma(\psi_i)) \equiv \lambda$. Therefore, $r_v(\sigma(\Delta_i)) \vDash_{ST^+} r_v(\sigma(\psi_i))$.

2B. $v(\sigma(\bigwedge \Delta_i)) = 1$ and $v(\sigma(\psi_i)) = \frac{1}{2}$.By Lemma 2, this means that $r_v(\sigma(\psi_i)) \equiv \lambda$. Therefore, $r_v(\sigma(\Delta_i)) \vDash_{ST^+} r_v(\sigma(\psi_i))$.

2C. $v(\sigma(\bigwedge \Delta_i)) = \frac{1}{2}$ and $v(\sigma(\psi_i)) = 0$. By Lemma 2, this means that for some $\delta \in \Delta_i$, $r_v(\sigma(\delta_i)) \equiv \lambda$. Therefore, $r_v(\sigma(\Delta_i)) \vDash_{ST^+} r_v (\sigma(\psi_i))$.

□

Now we can prove the main result. Suppose there is a replacement σ of the formulas $\delta \in \Delta_i$ and ψ_i by formulas of the object language, and a valuation v such that:
$v(\sigma(t(\Delta_1 \vDash \psi_1))) = 1$ or $\frac{1}{2}$, ... , $v(\sigma(t(\Delta_n \vDash \psi_k))) = 1$ or $\frac{1}{2}$, and
$v(\sigma(t(\Delta \vDash \psi))) = 0$.
In other words, $v(\sigma(\bigwedge \Delta_1 \supset \psi_1)) = 1$ or $\frac{1}{2}$, ... , $v(\sigma(\bigwedge \Delta_n \supset \psi_n)) = 1$ or $\frac{1}{2}$, and $v(\sigma(\bigwedge \Delta \supset \psi)) = 0$.

Now take the function r_v. By Lemma 4, $r_v(\sigma(\Delta_1)) \vDash_{ST^+} r_v(\sigma(\psi_1))$ and... and $r_v(\sigma(\Delta_n)) \vDash_{ST^+} r_v(\sigma(\psi_n))$. By Lemma 3, $r_v(\sigma(\Delta)) \nvDash_{ST^+} r_v(\sigma(\psi))$. r_v is a uniform and recursive function, so the composition $r_v \circ \sigma$ is also a uniform replacement function. Therefore, there are some formulas that can be replaced for $\delta \in \Delta_i$ and ψ_i that can be used as a counterexample for the metainference in ST^+. □

An example can illustrate this point. Suppose we are analyzing the failure of transitivity in ST^+ and its relation to the LP^+ conditional. The conditional of LP^+ is non-transitive. In other words, there are uniform replacements σ of A, B and C by formulas of the language such that $\sigma(A \supset B), \sigma(B \supset C) \nvDash_{LP^+} \sigma(A \supset C)$. For example, when $\sigma(A) = p, \sigma(B) = q \vee w, \sigma(C) = s$, we obtain a counterexample to the transitivity of the conditional, since $p \supset (q \vee w), (q \vee w) \supset s \nvDash_{LP^+} p \supset s$. This feature of LP^+ will correspond to the failure of transitivity in ST^+. To see this we take an LP^+ valuation v that witnesses this invalidity. Let v be a valuation such that $v(p) = 1$ and $v(s) = 0$, and $v(w) = 0$, so necessarily $v(q) = \frac{1}{2}$. Now $r(p) = \top$, $r(s) = \bot$, and $r(q \vee w) = (\lambda \vee \bot) \equiv \lambda$. It is easy to see that $\top \vDash_{ST^+} \lambda \vee \bot$, and $\lambda \vee \bot \vDash_{ST^+} \bot$; however $\top \nvDash \bot$. In other words, there are some uniform replacements σ of A, B and C by formulas of the language such that $\sigma(A) \vDash_{ST^+} \sigma(B), \sigma(B) \vDash_{ST^+} \sigma(C)$ but $\sigma(A) \nvDash_{ST^+} \sigma(C)$. Therefore, the failure of transitivity for LP^+'s conditional can be represented as the failure of transitivity for ST^+'s consequence relation.[10]

4 LP^+ and ST^+'s External Logic

In this section we will set up a proof procedure for both ST^+ and LP^+. Since these logics use three-valued models, it turns out that the task of comparing them proof-theoretically becomes much easier if they are presented by means of a three-sided sequent calculus.[11]

The idea of a two-sided sequent calculus is familiar enough. An expression of the form $\Gamma|\Delta$ means that the set of formulas Δ follows from the set of formulas Γ. If there are only two truth values, it is straightforward to provide a semantic reading for sequents of the form $\Gamma|\Delta$. We say that $\Gamma|\Delta$ holds if some formula in Δ has value 1 whenever all formulas in Γ have value 1. Equivalently, we can say that $\Gamma|\Delta$ holds if either some formula in Γ has value 0 or some formula in Δ has value 1. Three-sided sequents are of the form $\Gamma|\Sigma|\Delta$. The semantic reading of this is a generalization of the semantic reading for the two-sided sequents (bearing in mind that there is now a third semantic value $\frac{1}{2}$). The sequent $\Gamma|\Sigma|\Delta$ holds if either some formula in Γ has value 0 or some formula in Σ has value $\frac{1}{2}$ or some formula in Δ has value 1.[12]

Definition 7 (*The system $\mathcal{S}$*) Let A and B be any formulas, C an atomic formula and Γ, Σ, Δ, Γ', Σ', and Δ' any sets of formulas such that $\Gamma \subseteq \Gamma'$, $\Sigma \subseteq \Sigma'$, and $\Delta \subseteq \Delta'$. Also, for the $\forall$-rules, let t be any term, and a a variable not occurring in the rule's conclusion sequent. The proof system $\mathcal{S}$ has the following initial sequents and rules:

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[10] As an anonymous referee pointed out to us, a similar result applies to K_3^+. In particular, the First Collapse Result would still hold after replacing LP^+ by K_3^+ and ST^+ by TS^+ (where TS^+ is the tolerant-to-strict consequence relation, dual to ST^+).

[11] As far as we know, the first to give a precise formulation of multiple-sided sequents was G. Rousseau in [23]. A more modern and complete presentation can be found in the work of Baaz and others (see for example [3] and [2]) and in Paoli's book [18].

[12] In the literature it is common to find an alternative reading of three-sided sequents in terms of negative conjunctions. On that reading, $\Gamma|\Sigma|\Delta$ holds if it is not the case that: all members of Γ have value 1 and all members of Σ have value $\frac{1}{2}$ and all members of Δ have value 0. While for two-sided sequents these two readings are equivalent, this is not necessarily so for three-sided sequents. In fact, in [22] Ripley offers two proof procedures for ST^+, one based on the disjunctive reading and the other on the negative conjunctive reading. Although both systems characterize the same set of valid inferences, they are quite different in many respects. In any case, we think that the disjunctive reading is less cumbersome, and that's why we use it in this section. However, by appropriately modifying the corresponding definitions, a version of the result we obtain below can be proved for Ripley's other system as well (unfortunately, substantiating this claim would take too much space).

$$\text{Reflexivity}\ \frac{}{C|C|C}$$

$$\text{Cut}\ \frac{\Gamma,\text{A}|\Sigma,\text{A}|\Delta \qquad \Gamma,\text{A}|\Sigma|\Delta,\text{A} \qquad \Gamma|\Sigma,\text{A}|\Delta,\text{A}}{\Gamma|\Sigma|\Delta}$$

$$\text{Weakening}\ \frac{\Gamma|\Sigma|\Delta}{\Gamma'|\Sigma'|\Delta'}$$

$$\text{Left}\neg\ \frac{\Gamma|\Sigma|\Delta,\text{A}}{\Gamma,\neg\text{A}|\Sigma|\Delta}$$

$$\text{middle}\neg\ \frac{\Gamma|\Sigma,\text{A}|\Delta}{\Gamma|\Sigma,\neg\text{A}|\Delta}$$

$$\text{Right}\neg\ \frac{\Gamma,\text{A}|\Sigma|\Delta}{\Gamma|\Sigma|\Delta,\neg\text{A}}$$

$$\text{Left}\wedge\ \frac{\Gamma,\text{A},\text{B}|\Sigma|\Delta}{\Gamma,\text{A}\wedge\text{B}|\Sigma|\Delta}$$

$$\text{Middle}\wedge\ \frac{\Gamma|\Sigma,\text{A}\ |\Delta,\text{A} \qquad \Gamma|\Sigma,B|\Delta,\text{B} \qquad \Gamma|\Sigma,\text{A},\text{B}|\Delta}{\Gamma|\Sigma,\text{A}\ \wedge\text{B}\ |\Delta}$$

$$\text{Right}\wedge\ \frac{\Gamma|\Sigma|\Delta,\text{A} \qquad \Gamma|\Sigma|\Delta,\text{B}}{\Gamma|\Sigma|\Delta,\text{A}\wedge\text{B}}$$

$$\text{Left}\forall\ \frac{\Gamma,A(t)|\Sigma|\Delta}{\Gamma,\forall xA(x)|\Sigma|\Delta}$$

$$\text{Middle}\forall\ \frac{\Gamma|\Sigma,A(a)|\Delta,A(a) \qquad \Gamma|\Sigma,A(t)|\Delta}{\Gamma|\Sigma,\forall xA(x)|\Delta}$$

$$\text{Right}\forall\ \frac{\Gamma|\Sigma|\Delta,A(a)}{\Gamma|\Sigma|\Delta,\forall xA(x)}$$

$$\text{LeftT}\ \frac{\Gamma,A|\Sigma|\Delta}{\Gamma,T\langle A\rangle|\Sigma|\Delta}$$

$$\text{middleT}\ \frac{\Gamma|\Sigma,\text{A}|\Delta}{\Gamma|\Sigma,T\langle A\rangle|\Delta}$$

$$\text{RightT}\ \frac{\Gamma|\Sigma|\Delta,\text{A}}{\Gamma|\Sigma|\Delta,T\langle A\rangle}$$

A few remarks on the proof system are in order. First, in a two-sided sequent calculus the rule of Cut can be taken to express an exclusivity constraint on the formulas of the language. It tells us, roughly, that no formula can be true and false at the same time. In a three-sided sequent calculus, like the one we are considering, the rule of Cut also expresses an exclusivity constraint. But since there are now three truth-values, it tells us that no formula can receive more than one of the three truth-values, that is, no formula can be 1 and $\frac{1}{2}$, or $\frac{1}{2}$ and 0, or 1 and 0. This is usually captured by having three Cut rules in the system.[13]

However, the Cut rule above is simpler and it is equivalent to these three rules (see [22] for a proof that it can be derived from the three rules; that the three rules are derivable from it is easy to see).

But hold on! Wasn't $\mathcal{S}$ supposed to be a Cut-free system? Not exactly, $\mathcal{S}$ has Cut. This might be surprising, but it is as it should be. Even though Cut and transitivity amount to the same thing in a two-sided sequent calculus, this is not the case for three-sided sequents. In fact, it can be shown that Cut is not an admissible rule of $\mathcal{S}$.[14] The sequent $\varnothing|\lambda|\varnothing$ is provable using Cut but not without it (see again [22], Fact 4.2). Given that Cut is available in $\mathcal{S}$, the reader might be curious as to how the Liar paradox is blocked in this framework. With two-sided sequents, the absence of Cut prevents us from inferring the empty sequent from $\varnothing \Rightarrow \lambda$ and $\lambda \Rightarrow \varnothing$. In $\mathcal{S}$, the Liar paradox is prevented because the sequent $\lambda|\varnothing|\lambda$ is not provable. If it were, an application of Cut would deliver the empty sequent, since both $\lambda|\lambda|\varnothing$ and $\varnothing|\lambda|\lambda$ are provable.

Second, the Reflexivity axioms are formulated only for atomic formulas, but later on we will use the fact that they also hold for any formula. This is unproblematic. A straightforward induction on complexity shows that if Reflexivity holds for atomic formulas, it also holds for any formulas.

Third, there is an uninteresting difference between $\mathcal{S}$ and the system in [22]. Weakening is not officially a structural rule of the later system (rather it is built in). However, to make the proof of Theorem 4.1 below more clear we add it explicitly to the system $\mathcal{S}$.

[13]These rules have the form:

$$Cut_1\ \frac{\Gamma, A|\Sigma|\Delta \qquad \Gamma|\Sigma, A|\Delta}{\Gamma|\Sigma|\Delta}$$
$$Cut_2\ \frac{\Gamma|\Sigma, A|\Delta \qquad \Gamma|\Sigma|\Delta, A}{\Gamma|\Sigma|\Delta}$$
$$Cut_3\ \frac{\Gamma, A|\Sigma|\Delta \qquad \Gamma|\Sigma|\Delta, A}{\Gamma|\Sigma|\Delta}$$

[14]Interestingly, since $\mathcal{S}$ can be used to represent not only ST^+ but also K_3^+ and LP^+ (all that changes is the definition of validity, the structural and operational rules remain the same), Cut is not admissible for K_3^+ and LP^+ either. Cut does become admissible in the proof procedure based on the negative conjunctive reading. Thanks to an anonymous referee for bringing up this issue.

Fourth, we assume that Γ, Σ, Δ and so on, are *finite* sets of formulas. So every sequent of the form $\Gamma|\Sigma|\Delta$ can always be represented as $\gamma_1, ..., \gamma_n|\sigma_1, ..., \sigma_m|\delta_1, ..., \delta_j$.

Now we turn to the notion of proof-theoretic validity. The system is compatible with several definitions of proof-theoretic validity. Since we want to compare ST^+ with LP^+, it will be useful to define what counts as a valid argument in these logics.

Definition 8 (*Proof-theoretic validity for* ST^+) We say that an argument from the set Γ to the set Δ is *proof-theoretically-valid in* ST^+ (notationally, $\Gamma \vdash_{ST^+} \Delta$) if the sequent $\Gamma|\Gamma, \Delta|\Delta$ is provable in $\mathcal{S}$.

Definition 9 (*Proof-theoretic validity for* LP^+) We say that an argument from the set Γ to the set Δ is *proof-theoretically-valid in* LP^+ (notationally, $\Gamma \vdash_{LP^+} \Delta$) if the sequent $\Gamma|\Delta|\Delta$ is provable in $\mathcal{S}$.[15]

Observe that there is absolutely no difference regarding the rules and axioms of ST^+ and LP^+. The only difference between them has to do with the sort of sequent that should be derivable for an argument to be proof-theoretically valid.

At this point we have what we need to introduce the notion of external validity. Let's fix a language $\mathcal{L}$ and a certain theory $\mathcal{T}$ formulated by means of a (two-sided) sequent calculus $\mathcal{SC}$. For a set Γ of $\mathcal{L}$-formulas and a single $\mathcal{L}$-formula A, we say that Γ externally implies A in $\mathcal{T}$ whenever the sequent $\varnothing|$A is provable in the calculus obtained from $\mathcal{SC}$ by adding as initial sequents all sequents $\varnothing|\gamma$, for $\gamma \in \Gamma$. This is the notion of external validity for the theory $\mathcal{T}$.[16]

The problem is that this notion of external logic is not applicable to the theory ST^+ as we have presented it in this section. ST^+ was presented with the aid of the system $\mathcal{S}$ which, on the one hand, works with multiple conclusions and, on the other, has three-sided sequents. However, the problem can be easily solved by slightly modifying the characterization given above. This can be done in the following way:

Definition 10 (*External validity for* ST^+) We say that an argument from the set Γ to the set Δ is *externally valid in* ST^+ (notationally, $\Gamma \vdash_{ST^+_E} \Delta$) if the sequent $\varnothing|\Delta|\Delta$ is provable from $\mathcal{S}$ together with $\varnothing|\gamma|\gamma$ for every γ in Γ (i.e. from the system that results from adding these as initial sequents to $\mathcal{S}$).[17]

[15]Notice that we can also define proof-theoretic validity for K_3 and other three-valued logics using the system $\mathcal{S}$. For K_3 the definition would be straightforward. An argument from the set Γ to the set Δ is *proof-theoretically-valid in* K_3^+ (notationally, $\Gamma \vdash_{K_3^+} \Delta$) if the sequent $\Gamma|\Gamma|\Delta$ is provable in $\mathcal{S}$.

[16]A similar notion was introduced by Avron in [1] to study the properties of certain linear logics (the only difference being that he considers multisets instead of sets). More recently, Mares and Paoli use this notion in [15] to argue in favor of a substructural approach to paradoxes. Finally, Paoli has applied it more specifically to determine the strength of the logic ST^+ when it is formulated using a two-sided sequent calculus.

[17]Thanks to Francesco Paoli for this definition.

The reason for defining external validity in this way is that the provable formulas of ST^+ should be the formulas that receive in every model either the value 1 or the value $\frac{1}{2}$. There is no difference between ST^+ and LP^+ regarding the set of *formulas* each theory considers as valid. One immediate fact that follows from the previous definition is that for formulas (but not for inferences, as we will shortly see), ST^+'s external logic is the same as its internal logic which, of course, contains all classically valid formulas.

The previous definition is useful to compare $\vdash_{ST_E^+}$ with $\vdash_{LP^+}$ in yet another way. What the next theorem shows is that there is no real difference between $\vdash_{ST_E^+}$ and $\vdash_{LP^+}$, a result analogous to the one reached in the previous section.

Theorem 4 (Second Collapse Result) $\Gamma \vdash_{ST_E^+} \Delta$ *if and only if* $\Gamma \vdash_{LP^+} \Delta$.

Proof sketch Let Γ be the set $\{\gamma_1, ..., \gamma_n\}$. First we will prove the left-to-right direction. Assume that $\mathcal{S}$ together with the initial sequents $\varnothing|\gamma_i|\gamma_i$ for $1 \leq i \leq n$, proves $\varnothing|\Delta|\Delta$. Then there is a proof of $\varnothing|\Delta|\Delta$ that only uses the rules and axioms of $\mathcal{S}$ plus (possibly) the initial sequents $\varnothing|\gamma_1|\gamma_1, ..., \varnothing|\gamma_n|\gamma_n$. This proof can be easily transformed into an $\mathcal{S}$-proof of $\gamma_1, ..., \gamma_n|\Delta|\Delta$ in the following way. First take each sequent $\Gamma'|\Sigma'|\Delta'$ in the proof and add γ_i on the left side of the sequent for each γ_i in Γ, thus obtaining $\gamma_1, ..., \gamma_n, \Gamma'|\Sigma'|\Delta'$. Now the bottom sequent is no longer $\varnothing|\Delta|\Delta$ but $\gamma_1, ..., \gamma_n|\Delta|\Delta$, and it is clear that every step involving a connective rule is still valid (as the reader can check for herself), since none of the rules of $\mathcal{S}$ for the connectives becomes invalid by *uniformly* changing the context. This is also true in the case of the $\forall$-rules, but we need to make sure to change variables so that clashes are avoided. The resulting object may not be a proof of $\gamma_1, ..., \gamma_n|\Delta|\Delta$ because it may have tips of the form $\gamma_1, ..., \gamma_n, A|A|A$, or of the form $\gamma_1, ..., \gamma_n|\gamma_i|\gamma_i$, which are not initial sequents of $\mathcal{S}$. However, given that the rule of Weakening is in $\mathcal{S}$, we can take those tips and extend them upwards reaching sequents which are in fact initial sequents of $\mathcal{S}$. Now the resulting object is indeed an $\mathcal{S}$-proof of the sequent $\gamma_1, ..., \gamma_n|\Delta|\Delta$.

To prove the right-to-left direction assume that $\Gamma \vdash_{LP^+} \Delta$. Then there is an $\mathcal{S}$-proof of $\gamma_1, ..., \gamma_n|\Delta|\Delta$. We can transform this proof into a proof of the sequent $\varnothing|\Delta|\Delta$ from $\mathcal{S}$ together with the initial sequents $\varnothing|\gamma_1|\gamma_1, ..., \varnothing|\gamma_n|\gamma_n$. The idea is to cut the γ_i's in Γ one by one. For γ_1 we do the following:

$$\dfrac{\dfrac{\dfrac{}{\varnothing|\gamma_1|\gamma_1}\text{Axiom}}{\gamma_2,\ldots,\gamma_n|\gamma_1,\Delta|\gamma_1,\Delta}\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_1,\ldots,\gamma_n|\Delta|\Delta}}{\gamma_1,\ldots,\gamma_n|\gamma_1,\Delta|\Delta}\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_1,\ldots,\gamma_n|\Delta|\Delta}}{\gamma_1,\ldots,\gamma_n|\Delta|\gamma_1,\Delta}\text{Weak}}{\gamma_2,\ldots,\gamma_n|\Delta|\Delta}\text{Cut}$$

Once we have the sequent $\gamma_2, ..., \gamma_n|\Delta|\Delta$, the same can be done for γ_2:

$$\dfrac{\dfrac{\dfrac{}{\varnothing|\gamma_2|\gamma_2}\text{Axiom}}{\gamma_3,\ldots,\gamma_n|\gamma_2,\Delta|\gamma_2,\Delta}\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_2,\ldots,\gamma_n|\Delta|\Delta}}{\gamma_2,\ldots,\gamma_n|\gamma_2,\Delta|\Delta}\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_2,\ldots,\gamma_n|\Delta|\Delta}}{\gamma_2,\ldots,\gamma_n|\Delta|\gamma_2,\Delta}\text{Weak}}{\gamma_3,\ldots,\gamma_n|\Delta|\Delta}\text{Cut}$$

It should be clear that this procedure can be performed on each γ_i. The final part of the proof looks like this.

$$\dfrac{\dfrac{\dfrac{}{\varnothing\,|\,\gamma_n\,|\,\gamma_n}\,\text{Axiom}}{\varnothing\,|\,\gamma_n,\Delta\,|\,\gamma_n,\Delta}\,\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_n\,|\,\Delta\,|\,\Delta}}{\gamma_n\,|\,\gamma_n,\Delta\,|\,\Delta}\,\text{Weak} \qquad \dfrac{\dfrac{\vdots}{\gamma_n\,|\,\Delta\,|\,\Delta}}{\gamma_n\,|\,\Delta\,|\,\gamma_n,\Delta}\,\text{Weak}}{\varnothing\,|\,\Delta\,|\,\Delta}\,\text{Cut}$$

This completes the proof. □

So, again, it turns out that ST^+'s external logic, this time spelled out proof-theoretically, just amounts to the logic of LP^+.

There is a sense in which this result comes as no surprise. If the notion of external validity is exactly the same for LP^+ and ST^+, then it immediately follows that ST^+'s external logic is just LP^+, given that for non-substructural logics such as the ones we have been considering, external validity and internal validity amount to the same thing. In other words, it could be argued that the notion of external validity we introduced is tailored-made to be LP^+-validity.

However, we believe this claim is incorrect. While it is true that in principle there are other options for defining external validity for ST^+, none of them seem adequate. In particular, we could consider the following three alternative characterizations of external validity:

1. $\Gamma \vdash_{ST_E^+} \Delta$ if and only if the sequent $\varnothing|\varnothing|\Delta$ is provable from $\mathcal{S}$ together with $\varnothing|\varnothing|\gamma$ for each γ in Γ; or
2. $\Gamma \vdash_{ST_E^+} \Delta$ if and only if the sequent $\varnothing|\varnothing|\Delta$ is provable from $\mathcal{S}$ together with $\varnothing|\gamma|\gamma$ for each γ in Γ; or
3. $\Gamma \vdash_{ST_E^+} \Delta$ if and only if the sequent $\varnothing|\Delta|\Delta$ is provable from $\mathcal{S}$ together with $\varnothing|\varnothing|\gamma$ for each γ in Γ.

In the first case, $\vdash_{ST_E^+}$ is just K_3^+-validity, and in the second case, $\vdash_{ST_E^+}$ is just TS^+-validity.[18] The third case is more interesting. It can be proved that $\vdash_{ST_E^+}$ is just ST^+-validity, that is, ST^+'s external notion of validity would coincide with its internal notion of validity, a point that, we think, would be welcome by advocates of ST^+. The proofs of these facts are straightforward modifications of the proof of theorem 4.1.

Nevertheless, we think that none of these alternative characterizations of external validity are adequate for ST^+. The reason is that they do not capture what it is *for a formula* to be valid in ST^+. Recall that the notion of external validity works with formulas, not with arguments, and the valid formulas of ST^+ are simply the ones that have a value greater than 0 in every model.

The characterization we favor, on the other hand, captures this feature. More specifically, it rests on a natural generalization of the notion of external validity for two-sided sequents plus the fact that valid ST^+-formulas can take either the value 1 or the value $\frac{1}{2}$. Furthermore, there is independent evidence that we are getting things right. As we have seen, the logic of the metainferences of ST^+ is also the logic LP^+.

[18] We say that $\Gamma \vdash_{TS^+} \Delta$ if and only if $\Gamma|\varnothing|\Delta$ is provable in $\mathcal{S}$.

5 The Costs of Losing Transitivity

It seems prima facie desirable that anything claimed in the internal logic, should also be claimed in the “external” logic (whatever this external notion amounts to). The results of the previous sections seem to show that supporters of the non-transitive view do not care about this desideratum. In ST^+ the internal and the external notions of consequence are not the same. Either if we understand ‘external’ in the sense of the previous section (i.e. as ‘proof-theoretic external validity’) or if we understand it in the sense of section 3 (i.e. in terms of metainferences), ST^+’s external logic is weaker than its internal logic. In [9], the authors claim that Leitgeb’s desiderata in [14] concerning theories of truth are satisfied. They say (p. 864) that

> The argument that they cannot be jointly satisfied turns crucially on the assumption of transitivity, but transitivity is not among the eight desiderata, nor does it follow from them.

Of course, they are right, but it seems that when dealing with substructural theories, the utopic theory of truth envisioned by Leitgeb should include an extra requirement, namely that its internal and external logics coincide. From the arguments of the previous two sections is seem clear that this new desideratum is not satisfied by ST^+. Whether failing to fulfill this requirement is more costly than failing to fulfill the others is a matter that we will not settle here.

Now, it might argued that the lack of coincidence between the internal and the external consequence relations is not something the ST^+ theorist would be bothered by.

She could claim that the agreement between the external and the internal logic is just a classical desideratum (like, for example, the rule of Explosion). But it is important to see that it is not necessarily so. For not only classical logic, but also some non-classical systems such as intuitionistic logic, K_3, and LP fulfill this constraint as well. It is not a matter of classical vs. non-classical, but of structural vs. substructural, and as we know, most non-classical approaches in philosophical logic are structural.

There are many reasons why this desideratum is usually taken into account. For one thing, supporting a logic in which the external and the internal points of view do not coincide seems to be at least inconvenient, because the question ‘which logic do you support?’ has now an ambiguous answer. Moreover, it brings up the issue of the universality of logic: the question ‘which logic does it right?’ cannot have only one right response anymore. Of course, a pluralist view about logic could be put forward to deal with this difficulty. In fact, a quite elaborate response has been developed by Mares and Paoli [15], who argue that classical logic is an ambiguous logic: it not only fails to take into account the distinction between intensional and extensional connectives, but also the distinction between external and internal consequence. According to them, as a result of this ambiguity, semantic paradoxes are nothing more than fallacies of equivocation. However, although this sort of response seems to go well with an approach were the rules of Structural Contraction and Structural Weakening do not hold in general, it is hard to see how it can be adapted for a Cut-free approach.

Still, as an anonymous referee suggested to us, the ST^+ theorist might be skeptical about the importance of this problem for her main project, which is giving a theory of truth. But we want to stress that the external validity relation is as a matter of fact very relevant for the theory of truth. Therefore, the problem is not just with the disagreement between the external and the internal consequence relations, but with the weakness of the external notion (which we proved to be equivalent to LP^+).

It is usually acknowledged that one of the roles (if not the only role) fulfilled by the truth predicate is that of allowing the expression of certain kinds of generalizations that would otherwise be inexpressible. And we not only want to express such generalizations, but we also want to have a theory powerful enough to reason with them.

For example, sometimes we would like to endorse everything someone said even if we do not know exactly what she said, and on some occassions we would like to endorse a theory that has infinitely many theorems. In such cases, we usually appeal to truth generalizations, which have this form:

$$\forall x(\phi(x) \rightarrow Tr(x))$$

($\phi(x)$ might mean 'x is a theorem of arithmetic', 'the Pope said x', etc.)

But these generalizations can be effective just when the following inference is valid:

$$\forall x(\phi(x) \rightarrow Tr(x)), \phi\langle\psi\rangle \vDash \psi.$$

Given the failure of Modus Ponens in LP^+, it does not validate this inference. Therefore, we can say that LP^+ cannot represent one of the main features of the truth predicate.[19] Given what we have proved in the previous sections, an analogous difficulty affects ST^+. The reason is that the failure of *Meta* Modus Ponens yields statements such that

$$\vDash_{ST^+} \forall x(\phi(x) \rightarrow Tr(x))$$

and

$$\vDash_{ST^+} \phi\langle\psi\rangle$$

are both the case, but

$$\vDash_{ST^+} \psi$$

is not. There are situations where the first two claims are (tolerantly) assertible, but the third one is not. So we have cases where the truth predicate is not fulfilling its role as a device for truth generalizations, even though it is fully transparent.[20]

Finally, the previous results also shed light on another interesting aspect of ST^+ which has not been considered (as far as we know) in the literature: the meaning of ST^+'s logical expressions. Inferentialism can be identified roughly with the claim

[19] In order to cope with this and related problems, some supporters of LP-based theories of truth introduce a stronger conditional which validates Modus Ponens.

[20] Thanks to Lavinia Picollo and Thomas Schindler for this argument.

that the meaning of a logical expression is given by the rules governing its use.[21] Since the proof-theoretically valid inferences according to $\vdash_{ST^+}$ extend those of classical logic, one would expect $\wedge$, $\neg$, and so on, to be just the classical connectives, at least if one thinks that the meaning of a logical expression is specified by the class of provable sequents containing that expression. However, it is usually acknowledged that this criterion for sameness of meaning is problematic, given that it renders many logical disputes as verbal. A different criterion is given by what is usually called *operational minimalism* (see for instance [17] and [12]), where the meaning of a logical expression is fully specified by the operational rules for that expression. The notion of operational meaning is introduced to give a stable "core" meaning for logical constants over a range of systems. Then, according to this approach, classical logic and ST^+ determine the same meaning for logical constants given that both systems have the same operational rules. This idea is attractive, but it becomes quite hard to apply when substructural logics are considered (see [11]). The results in the previous sections point to a third idea: basically, the meaning of a logical expression is (at least partly) given by the metainferences (or external inferences) involving that expression. So to determine the meaning of a logical expression it is necessary to consider not only its operational rules but also the lines and the sequent arrow in the metainferences. But as far as metainferences are concerned, classical logic and ST^+ come apart. Hence, under this criterion, it is no longer clear that the logical expressions in ST^+ are the logical expressions in classical logic. The set of classically derivable sequents is closed under certain rules under which the set of ST^+-derivable sequents is not closed under. For example, in some cases (i.e. some replacement of A and B by formulas) while $A \vDash_{ST^+} B \wedge \neg B$, we have $\nvDash_{ST^+} \neg A$. But this does hold if the logical expressions involved are the classical ones. For another example, this time involving the external logic, having $\vdash_{ST^+} A$ and $\vdash_{ST^+} A \supset B$ does not give us $\vdash_{ST^+} B$. But again, it does if the conditional used there is the material conditional of classical logic.

To put the point a bit differently, we have shown that there is a strong disanalogy between classical logic and ST^+. Both the logic of the metainferences of ST^+ and its proof-theoretical external logic are very different from classical logic. These facts should impinge in some way on the meaning of the logical expressions of ST^+.

6 Conclusion

Supporters of ST^+ claim that this theory respects classical logic. They claim that the only difference with classical logic is that the consequence relation they propose is non-transitive. We have shown that, in a way, this non-transitive consequence relation bears a striking resemblance to LP^+'s conditional. Therefore, on the one hand, adopting ST^+ as a solution to the paradoxes is no more illuminating than adopting LP^+, because all the problems usually attributed to LP^+ will also affect ST^+

[21] Of course, inferentialism can take many forms, but for now we will work with this vague characterization.

external notion of consequence. On the other hand, there is a sense in which LP^+ is better, because there is no mismatch between its internal and its external consequence relations.

Acknowledgments We owe thanks to Norbert Gratzl, Ole Hjortland, Francesco Paoli, Lavinia Picollo, Dave Ripley, Thomas Schindler, and Johannes Stern for very helpful comments on previous versions of this paper. Some of this material was presented at conferences in Barcelona (Logos), Munich (MCMP) and Buenos Aires (Buenos Aires Logic Group). We are very grateful to the members of these audiences for their valuable feedback. We are also specially grateful to Jose Martinez and Elia Zardini for organizing the Substructural Approaches to Paradox Workshop and to Thomas Meier for organizing the 1st MCMP Munich-Buenos Aires Workshop.

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